Polynomial-phase estimation, phase unwrapping and the nearest lattice point problem

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Abstract—Polynomial-phase signals have attracted significant interest due to their applicability to radar, sonar, geophysics, and radio communication. In this paper we introduce a new technique for estimating the parameters of polynomial phase signals. The parameters are estimated by performing phase unwrapping in a least squares manner. The least squares problem is formulated as a nearest lattice point problem that can be solved using existing techniques. The statistical performance of the new estimator is excellent when compared with popular estimators such as those based on the discrete polynomial phase transform.

Index Terms—Polynomial-phase signals, lattice theory, parameter estimation, chirp signals, polynomials, radar signal processing, closest point search, lattice decoding.

I. INTRODUCTION

Polynomial-phase signals have numerous applications including radar, sonar, geophysics, and radio communication [1]. A uniformly sampled, constant amplitude, polynomial-phase signal of order \(m\) has the model [2],

\[
z_n = A \exp \left(2\pi j \sum_{k=0}^{m} p_k n^k \right) \tag{1}
\]

where \(n = 1, 2, \ldots, N\), the \(p_k\) are the polynomial phase parameters, \(A\) is the signal amplitude and \(j = \sqrt{-1}\). We will often write the parameters, \(p_k\), as a column vector \(p\). Of significant practical importance is the estimation of \(p\) from the signal \(y_n = z_n + s_n\) where the sequence \(s_1, s_2, \ldots, s_N\) is a complex noise process. Many estimators for polynomial-phase parameters have been studied and implemented [1, 3–8].

A popular method of estimation is by way of the discrete polynomial-phase transform (DPT) studied by Peleg et al. [2, 3, 9]. The DPT enables each parameter to be estimated iteratively using the fast Fourier transform in a way analogous to the frequency estimator of Rife and Boorstyn [10, 11]. Numerous modifications of the DPT estimator have been described. Two examples are O’Shea [5] and Golden and Friedlander [12]. These modifications trade computational performance and statistical accuracy.

Another estimation method is phase unwrapping. Phase unwrapping estimators appear to have been first suggested by Tretter [13]. Tretter focused on the specific case of frequency estimation i.e. when \(m = 1\). Tretter noticed that the phase of a complex sinusoid is a linear function that is ‘wrapped’ modulo \(2\pi\). If the phase could be ‘unwrapped’ then the parameters can be estimated by linear regression. An analogous situation occurs when \(m \neq 1\) by replacing the linear function with a polynomial of order \(m\). Many phase unwrapping estimators have been suggested for the specific cases when \(m = 1\) [14, 15], \(m = 2\) [4] and for the general case of polynomial phase estimation [7, 16].

A common approach to phase unwrapping is to compute the \(m\)th other difference of the complex argument of the \(y_n\). The resulting signal then resembles a moving average process and can be estimated by standard linear techniques. This approach was first suggested by Kay for the case when \(m = 1\) [14] and extended to the general case by Kitchen [7]. A significant advantage of this approach is that the moving average process has structure enough that the estimates can be computed with only \(O(N)\) arithmetic computations. The estimators also appear to be statistically efficient when the signal to noise ratio (SNR) is sufficiently large and \(N\) is sufficiently small.

An important consideration in polynomial-phase estimation is that of parameter aliasing. For example, consider the case when \(m = 0\). The model becomes

\[
y_n = A \exp (2\pi j p_0) + s_n
\]

where \(p_0 \in \mathbb{R}\). Then \(y_n\) is identical for \(p_0 + c\) for any \(c \in \mathbb{Z}\). In order to avoid these ambiguities we must restrict \(p_0\) to some interval of length 1. A natural choice is \([-1/2, 1/2)\). We call this the identifiable region (IR). For the case when \(m = 1\) we find that the IR is \([p_0, p_1] \in [-1/2, 1/2]^2\) which corresponds with the Nyquist criterion. In [17] it is shown that, for larger \(m\), the IR can be represented as the rectangular prism \(\prod_{k=0}^{m} [-0.5/k, 0.5/k]\). Moreover, estimators based on the DPT only operate correctly over a small fraction of the IR.

In this paper we consider a new phase unwrapping estimator for polynomial-phase parameters. The approach is to unwrap the phase in a least squares manner. We call the resulting estimator the least squares phase unwrapping estimator (LSU). When \(m = 0\) and when \(m = 1\) the LSU is known to be strongly consistent and its central limit theorem has been derived [18, 19]. For the case when \(m = 1\), Clarkson showed
of squares function
\[
SS(\hat{p}, u) = \sum_{n=1}^{N} \left( x_n - u_n - \sum_{k=0}^{m} \hat{p}_k n^k \right)^2.
\]
where we write the \( u_n \) in vector form as \( u \). The LSU then returns
\[
\hat{p} = \arg \min_{p \in IR} \min_{u \in \mathbb{Z}^N} SS(\hat{p}, u)
\]
where IR is the identifiable region described in [17].

III. LATTICE THEORY

A lattice, \( L \), is a set of points in \( \mathbb{R}^n \) such that
\[
L = \{ x \in \mathbb{R}^n \mid x = Bw, w \in \mathbb{Z}^n \}
\]
where \( B \) is called the generator or basis matrix [26]. A fundamental problem in lattice theory is the nearest lattice point problem. Given \( y \in \mathbb{R}^n \) and some lattice \( L \) whose lattice points lie in \( \mathbb{R}^n \), the nearest lattice point problem is to find the lattice point \( x \in L \) such that the Euclidean distance between \( y \) and \( x \) is minimised. We will use the notation NP\( \cap(Y, L) \) to denote the nearest point in \( L \) to \( y \).

The nearest lattice point problem is known to be NP-hard under certain conditions when the lattice itself is considered as an additional input parameter [27, 28]. Nevertheless, algorithms exist that can compute the nearest lattice point in reasonable time if the dimension is small [20–22]. One such algorithm introduced by Pohst [21] in 1981 was popularised in signal processing and communications fields by Viterbo and Boutros [22] and has since been called the sphere decoder. Approximate algorithms for computing the nearest point have also been studied. One example is Babai’s nearest plane algorithm [23] which has worst case polynomial complexity.

IV. LEAST SQUARES UNWRAPPING AND THE NEAREST LATTICE POINT PROBLEM

In this section we show how the LSU can be represented as a nearest lattice point problem in a lattice determined by \( N \) and \( m \). If we let \( n^k = [1^k, 2^k, 3^k, \ldots, n^k] \) where \( \cdot \) denotes transpose and define the Vandermonde matrix
\[
M = [n_0^n \ 1 \ \ldots \ n^{m-1} n^m]
\]
then \( SS(\hat{p}, u) \) may be written in vector form as
\[
SS(\hat{p}, u) = \| y - u - Mp \|^2. \tag{5}
\]
Given \( u \) the least squares estimator of \( p \) is obtained by the usual linear regression formulae and is given by
\[
\hat{p} = M^+ (y - u) \tag{6}
\]
where \( M^+ = (M'M)^{-1}M' \). Substituting (6) into (5) and rearranging, the least squares estimate of \( u \), conditioned on maximisation with respect to \( p \), is
\[
\hat{u} = \arg \min_{u \in \mathbb{Z}^N} \| B(y - u) \|. \tag{7}
\]
where \( B = I - MM^+ \) and \( I \) is the \( N \times N \) identity matrix.
Let \( \Lambda \) be the lattice with generator matrix \( \mathbf{B} \). Then the least squares phase unwrapping is the \( \hat{u} \) such that \( \mathbf{B}\hat{u} \) is the nearest lattice point in \( \Lambda \) to \( \mathbf{By} \). The least squares estimate of \( \mathbf{p} \) (6) can be computed by first computing \( \text{NPt}(\mathbf{By}, \Lambda) \), producing both \( \mathbf{B}\hat{u} \) and \( \hat{u} \). \( \mathbf{p} \) can then be computed using (6). After this procedure it is possible that the \( \hat{p} \) obtained is not in the IR but instead is an aliased version of the desired estimate. This aliasing can be resolved using the procedure described in [17].

The most difficult part of this procedure is finding the nearest lattice point. When \( m = 0 \) the lattice \( \Lambda \) coincides with the well studied lattice \( A_N^{N-1} \) [26, 29–31] for which a linear-time nearest point algorithm exists [32]. When \( m = 1 \) an algorithm to compute the nearest point in \( O(N^3 \log N) \) operations is known [19]. For \( m > 1 \) we propose two solutions. Firstly, the sphere decoder can be used to compute the nearest point and return the true LSU. Unfortunately the sphere decoder has worst case exponential complexity and therefore is only computationally feasible when \( N \) is small. An alternative approach is to compute an approximate nearest point using Babai’s nearest plane algorithm [23]. The estimate is no longer guaranteed to correspond to the LSU but the nearest point can be computed with \( O(N^2) \) arithmetic operations. Notice that Babai’s algorithm requires \( \Omega(N^2) \) operations. This requires \( O(N^4) \) operations. However, in this case, LLL reduction can be performed offline and the remainder of Babai’s algorithm requires \( O(N^2) \) operations.

V. Simulations

In this section we evaluate the statistical performance of the estimators using Monte Carlo simulation. We compare the LSU calculated exactly using the sphere decoder and calculated approximately using Babai’s algorithm with Kitchen’s phase unwrapping estimator [7] and Peleg et al.’s estimator based on the DPT [3]. The simulations are run with \( m = 3 \) and \( N = 50 \) under the assumption that the \( s_n \) are independent and identically distributed complex Gaussian random variables with real and imaginary parts that have variance \( \sigma^2 \). The mean square error (MSE) of the estimator after 1000 trials is computed for each value of \( SNR = A/2\sigma^2 \) in the range [2 dB, 20 dB]. The MSE is calculated modulo the IR as described in [17].

In Figure 1, \( \mathbf{p} = [0.1, 0.05, 0.01, 0.0005] \). This is within the range that is suitable for the DPT estimator [3, 17]. It is evident that the sphere decoder estimator is statistically superior to the other estimators. A rather significant penalty is paid by using Babai’s approximate nearest point algorithm.

In Figure 2 the parameters used are randomly generated so that they are uniformly distributed in the IR. It was noted in [17] that the DPT estimator only operates correctly on a small subset of the IR. The result of this is that the DPT estimator performs very poorly for parameters uniformly distributed in the IR. This effect is clearly seen in Figure 2. It appears that Kitchen’s estimator suffers from a similar problem. By comparison, both the sphere decoder and Babai’s LSU are unaffected.

VI. Conclusion

The paper introduces a new technique for estimating the parameters of a polynomial phase signal. The method is based on least squares phase unwrapping. It is found that the LSU estimator can be computed by solving the nearest lattice point problem in a lattice determined by \( N \) and \( m \). The estimator appears to have good statistical performance when compared with the DPT estimator and Kitchen’s estimator. The estimator also performs equally well over the entire IR. Best performance is obtained when the LSU is used in conjunction with the sphere decoder. A significant disadvantage is that this is exponential complex in \( N \). A compromise is to use Babai’s algorithm which has \( O(N^2) \) and quite good statistical performance. It may be that faster nearest point algorithms exist for these lattices. This is a topic for future research.

REFERENCES

Fig. 1. MSE in each of the four polynomial parameters with $N = 50$ and $p = [0.1, 0.05, 0.01, 0.0005]$.

Fig. 2. MSE in each of the four polynomial parameters with $N = 50$ and $p$ uniformly distributed in $R$. 