Polynomial-phase estimation, phase unwrapping and the nearest lattice point problem

Robby G. McKilliam[1], I. Vaughan L. Clarkson[1], Barry G. Quinn[2] and Bill Moran[3]

[1]School of Information Technology & Electrical Engineering, The University of Queensland, QLD 4067, AUSTRALIA
{robertm,v.clarkson}@itee.uq.edu.au

[2]Department of Statistics, Macquarie University, Sydney, NSW, 2109, Australia

[3]Department of Electrical Engineering & Computer Science, Melbourne Systems Lab, Dept of Elec & Electronic Eng, Uni of Melbourne, Vic. 3010, AUSTRALIA.

***Abstract*—Polynomial-phase signals have attracted significant interest due to their applicability to radar, sonar, geophysics, and radio communication. In this paper we introduce a new technique for estimating the parameters of polynomial phase signals. The parameters are estimated by performing *phase unwrapping* in a least squares manner. The least squares problem is formulated as a *nearest lattice point problem* that can be solved using existing techniques. The statistical performance of the new estimator is excellent when compared with popular estimators such as those based on the discrete polynomial phase transform.**

***Index Terms*—Polynomial-phase signals, lattice theory, parameter estimation, chirp signals, polynomials, radar signal processing, closest point search, lattice decoding.**

I. INTRODUCTION

Polynomial-phase signals have numerous applications including radar, sonar, geophysics, and radio communication [1]. A uniformly sampled, constant amplitude, polynomial-phase signal of order m has the model [2],

$$z_n = A \exp\left(2\pi j \sum_{k=0}^{m} p_k n^k\right) \tag{1}$$

where $n = 1, 2 \dots, N$, the p_k are the polynomial phase parameters, A is the signal amplitude and $j = \sqrt{-1}$. We will often write the parameters, p_k, as a column vector $\mathbf{p}$. Of significant practical importance is the estimation of $\mathbf{p}$ from the signal $y_n = z_n + s_n$ where the sequence $s_1, s_2, \dots, s_N$ is a complex noise process. Many estimators for polynomial-phase parameters have been studied and implemented [1, 3–8].

A popular method of estimation is by way of the discrete polynomial-phase transform (DPT) studied by Peleg *et. al.* [2, 3, 9]. The DPT enables each parameter to be be estimated iteratively using the fast Fourier transform in a way analogous to the frequency estimator of Rife and Boorstyn [10, 11]. Numerous modifications of the DPT estimator have been described. Two examples are O'Shea [5] and Golden and Friedlander [12]. These modifications trade computational performance and statistical accuracy.

Another estimation method is phase unwrapping. Phase unwrapping estimators appear to have been first suggested by Tretter [13]. Tretter focused on the specific case of frequency estimation i.e. when $m = 1$. Tretter noticed that the phase of a complex sinusoid is a linear function that is 'wrapped' modulo 2π. If the phase could be 'unwrapped' then the parameters can be estimated by linear regression. An analogous situation occurs when $m \neq 1$ by replacing the linear function with a polynomial of order m. Many phase unwrapping estimators have been suggested for the specific cases when $m = 1$ [14, 15], $m = 2$ [4] and for the general case of polynomial phase estimation [7, 16].

A common approach to phase unwrapping is to compute the mth other difference of the complex argument of the y_n. The resulting signal then resembles a moving average process and can be estimated by standard linear techniques. This approach was first suggested by Kay for the case when $m = 1$ [14] and extended to the general case by Kitchen [7]. A significant advantage of this approach is that the moving average process has structure enough that the estimates can be computed with only $O(N)$ arithmetic computations. The estimators also appear to be statistically efficient when the signal to noise ratio (SNR) is sufficiently large and N is sufficiently small.

An important consideration in polynomial-phase estimation is that of parameter aliasing. For example, consider the case when $m = 0$. The model becomes

$$y_n = A \exp(2\pi j p_0) + s_n$$

where $p_0 \in \mathbb{R}$. Then y_n is identical for $p_0 + c$ for any $c \in \mathbb{Z}$. In order to avoid these ambiguities we must restrict p_0 to some interval of length 1. A natural choice is $[-1/2, 1/2)$. We call this the identifiable region (IR). For the case when $m = 1$ we find that the IR is $(p_0, p_1) \in [-1/2, 1/2)^2$ which corresponds with the Nyquist criterion. In [17] it is shown that, for larger m, the IR can be represented as the rectangular prism $\prod_{k=0}^{m} [-0.5/k!, 0.5/k!)$. Moreover, estimators based on the DPT only operate correctly over a small fraction of the IR.

In this paper we consider a new phase unwrapping estimator for polynomial-phase parameters. The approach is to unwrap the phase in a least squares manner. We call the resulting estimator the least squares phase unwrapping estimator (LSU). When $m = 0$ and when $m = 1$ the LSU is known to be strongly consistent and its central limit theorem has been derived [18, 19]. For the case when $m = 1$, Clarkson showed how the LSU is related to a problem in algorithmic number theory known as the nearest lattice point problem [15, 20]. We extend this result to the general case in this paper. We show how the LSU can be computed by solving the nearest lattice point problem in a lattice determined by N and m.

The paper is organized as follows. In Section II we describe the objective function used by the LSU. In Section III we introduce some basic concepts in lattice theory including the nearest lattice point problem. We introduce two common techniques for solving the nearest lattice point problem, the sphere decoder [20–22] and Babai's nearest plane algorithm [23]. In Section IV we show how the LSU can be represented as a nearest lattice point problem. By using the sphere decoder the exact LSU can be found. By using Babai's algorithm an approximation of the LSU can be made. In Section V we use Monte Carlo simulation to analyse the statistical performance of the LSU, the phase unwrapping estimator of Kitchen [7] and the DPT estimator of Peleg [3]. We find that the LSU is statistically superior to the other estimators. Moreover, in contrast to the other estimators, we find that the LSU performs well for parameters anywhere in the IR.

II. Least squares phase unwrapping

The complex argument of the y_n, denoted $\angle y_n$, is given by

$$\angle y_n = 2\pi \left(v_n + \sum_{k=0}^{m} p_k n^k \right) \pmod{2\pi},$$

where v_n is noise in the range $[-1/2, 1/2)$ and has a distribution depending only on A and the probability density function (pdf) of the s_n. When s_n is complex Gaussian noise the distribution of the v_n is known as the projected normal distribution and has been studied by Mardia and Jupp [24, p. 46]. The distribution has also been discussed by Quinn [18] and Tretter [13]. Other circular noise distributions may be used, for example, the wrapped normal, or von Mises distributions [24, 25]. In this paper we make the assumption that the v_n are zero mean, independent and identically distributed. We also assume that the probability density function of the v_n is symetric and unimodal with mode at 0.

Let $x_n = \angle y_n / (2\pi)$. Then

$$x_n = \left\{ v_n + \sum_{k=0}^{m} p_k n^k \right\}, \tag{2}$$

where $\{x\} = x - \lfloor x \rceil$ is the *centered* fractional part of x and $\lfloor x \rceil$ denotes the nearest integer to x.[1]

We may write (2) as

$$x_n = v_n + u_n + \sum_{k=0}^{m} p_k n^k, \tag{3}$$

where $u_n = -\left\lfloor v_n + \sum_{k=0}^{m} p_k n^k \right\rceil$. By considering the u_n as nuisance parameters, where $u_n \in \mathbb{Z}$, we may derive the sum of squares function

$$SS(\tilde{\mathbf{p}}, \mathbf{u}) = \sum_{n=1}^{N} \left(x_n - u_n - \sum_{k=0}^{m} \tilde{p}_k n^k \right)^2. \tag{4}$$

where we write the u_n in vector form as $\mathbf{u}$. The LSU then returns

$$\hat{\mathbf{p}} = \arg \min_{\tilde{\mathbf{p}} \in \mathrm{IR}} \min_{\mathbf{u} \in \mathbb{Z}^N} SS(\tilde{\mathbf{p}}, \mathbf{u})$$

where IR is the identifiable region described in [17].

III. Lattice theory

A lattice, L, is a set of points in $\mathbb{R}^n$ such that

$$L = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{B}\mathbf{w}, \mathbf{w} \in \mathbb{Z}^n\}$$

where $\mathbf{B}$ is called the *generator* or *basis* matrix [26]. A fundamental problem in lattice theory is the *nearest lattice point problem*. Given $\mathbf{y} \in \mathbb{R}^n$ and some lattice L whose lattice points lie in $\mathbb{R}^n$, the nearest lattice point problem is to find the lattice point $\mathbf{x} \in L$ such that the Euclidean distance between $\mathbf{y}$ and $\mathbf{x}$ is minimised. We will use the notation $\mathrm{NPt}(\mathbf{y}, L)$ to denote the nearest point in L to $\mathbf{y}$.

The nearest lattice point problem is known to be NP-hard under certain conditions when the lattice itself is considered as an additional input parameter [27, 28]. Nevertheless, algorithms exist that can compute the nearest lattice point in reasonable time if the dimension is small [20–22]. One such algorithm introduced by Pohst [21] in 1981 was popularised in signal processing and communications fields by Viterbo and Boutros [22] and has since been called the *sphere decoder*. Approximate algorithms for computing the nearest point have also been studied. One example is Babai's nearest plane algorithm [23] which has worst case polynomial complexity.

IV. Least squares unwrapping and the nearest lattice point problem

In this section we show how the LSU can be represented as a nearest lattice point problem in a lattice determined by N and m. If we let $\mathbf{n}^k = [1^k, 2^k, 3^k, \dots, n^k]'$ where $'$ denotes transpose and define the Vandermonde matrix

$$\mathbf{M} = [\mathbf{n}^0 \ \mathbf{n}^1 \ \dots \ \mathbf{n}^{m-1} \ \mathbf{n}^m]$$

then $SS(\tilde{\mathbf{p}}, \mathbf{u})$ may be written in vector form as

$$SS(\tilde{\mathbf{p}}, \mathbf{u}) = \|\mathbf{y} - \mathbf{u} - \mathbf{M}\tilde{\mathbf{p}}\|^2. \tag{5}$$

Given $\mathbf{u}$ the least squares estimator of $\mathbf{p}$ is obtained by the usual linear regression formulae and is given by

$$\hat{\mathbf{p}} = \mathbf{M}^{+} (\mathbf{y} - \mathbf{u}) \tag{6}$$

where $\mathbf{M}^{+} = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'$. Substituting (6) into (5) and rearranging, the least squares estimate of $\mathbf{u}$, conditioned on maximisation with respect to $\tilde{\mathbf{p}}$, is

$$\hat{\mathbf{u}} = \arg \min_{\mathbf{u} \in \mathbb{Z}^N} \|\mathbf{B}(\mathbf{y} - \mathbf{u})\|. \tag{7}$$

where $\mathbf{B} = \mathbf{I} - \mathbf{M}\mathbf{M}^{+}$ and $\mathbf{I}$ is the $N \times N$ identity matrix.

[1] The direction of rounding for half-integers is not important, so long as it is consistent. The authors have chosen to round up half-integers here.

Let Λ be the lattice with generator matrix $\mathbf{B}$. Then the least squares phase unwrapping is the $\hat{\mathbf{u}}$ such that $\mathbf{B}\hat{\mathbf{u}}$ is the nearest lattice point in Λ to $\mathbf{By}$. The least squares estimate of $\mathbf{p}$ (6) can be computed by first computing $\mathrm{NPt}(\mathbf{By}, \Lambda)$, producing both $\mathbf{B}\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}$. $\hat{\mathbf{p}}$ can then be computed using (6). After this procedure it is possible that the $\hat{\mathbf{p}}$ obtained is not in the IR but instead is an aliased version of the desired estimate. This aliasing can be resolved using the procedure described in [17].

The most difficult part of this procedure is finding the nearest lattice point. When $m = 0$ the lattice Λ coincides with the well studied lattice A_{N-1}^* [26, 29–31] for which a linear-time nearest point algorithm exists [32]. When $m = 1$ an algorithm to compute the nearest point in $O(N^3 \log N)$ operations is known [19]. For $m > 1$ we propose two solutions. Firstly, the sphere decoder can be used to compute the nearest point and return the true LSU. Unfortunately the sphere decoder has worst case exponential complexity and therefore is only computationally feasible when N is small. An alternative approach is to compute an approximate nearest point using Babai's nearest plane algorithm [23]. The estimate is no longer guaranteed to correspond to the LSU but the nearest point can be computed with $O(N^2)$ arithmetic operations. Notice that Babai's algorithm requires LLL lattice reduction [33]. This requires $O(N^4)$ operations. However, in this case, LLL reduction can be performed offline and the remainder of Babai's algorithm requires $O(N^2)$ operations.

V. Simulations

In this section we evaluate the statistical performance of the estimators using Monte Carlo simulation. We compare the LSU calculated exactly using the sphere decoder and calculated approximately using Babai's algorithm with Kitchen's phase unwrapping estimator [7] and Peleg *et al.*'s estimator based on the DPT [3]. The simulations are run with $m = 3$ and $N = 50$ under the assumption that the s_n are independent and identically distributed complex Gaussian random variables with real and imaginary parts that have variance σ^2. The mean square error (MSE) of the estimator after 1000 trials is computed for each value of $SNR = A/2\sigma^2$ in the range $[2\,\mathrm{dB}, 20\,\mathrm{dB}]$. The MSE is calculated modulo the IR as described in [17].

In Figure 1, $\mathbf{p} = [0.1, 0.05, 0.01, 0.0005]$. This is within the range that is suitable for the DPT estimator [3, 17]. It is evident that the sphere decoder estimator is statistically superior to the other estimators. A rather significant penalty is paid by using Babai's approximate nearest point algorithm.

In Figure 2 the parameters used are randomly generated so that they are uniformly distributed in the IR. It was noted in [17] that the DPT estimator only operates correctly on a small subset of the IR. The result of this is that the DPT estimator performs very poorly for parameters uniformly distributed in the IR. This effect is clearly seen in Figure 2. It appears that Kitchen's estimator suffers from a similar problem. By comparison, both the sphere decoder and Babai's LSU are unaffected.

VI. Conclusion

The paper introduces a new technique for estimating the parameters of a polynomial phase signal. The method is based on least squares phase unwrapping. It is found that the LSU estimator can be computed by solving the nearest lattice point problem in a lattice determined by N and m. The estimator appears to have good statistical performance when compared with the DPT estimator and Kitchen's estimator. The estimator also performs equally well over the entire IR. Best performance is obtained when the LSU is used in conjunction with the sphere decoder. A significant disadvantage is that this is exponential complex in N. A compromise is to use Babai's algorithm which has $O(N^2)$ and quite good statistical performance. It may be that faster nearest point algorithms exist for these lattices. This is a topic for future research.

References

[1] J. Ängeby, "Estimating signal parameters using the nonlinear instantaneous least squares approach," *IEEE Trans. Signal Process.*, vol. 48, pp. 2721–2732, Oct. 2000.

[2] S. Peleg and B. Porat, "Estimation and classification of polynomial-phase signals," *IEEE Trans. Inform. Theory*, vol. 37, no. 2, pp. 422–430, Mar. 1991.

[3] S. Peleg and B. Friedlander, "The discrete polynomial-phase transform," *IEEE Trans. Signal Process.*, vol. 43, no. 8, pp. 1901–1914, Aug. 1995.

[4] P. M. Djuric and S. M. Kay, "Parameter estimation of chirp signals," *IEEE Trans. Acoust. Speech Signal Process.*, vol. 38, no. 12, pp. 2118–2126, Dec. 1990.

[5] P. J. O'Shea, "An iterative algorithm for estimating the parameters of polynomial phase signals," in *Proc. Internat. Sympos. Signal Process. Appl.*, 1996, vol. 2, pp. 730–731.

[6] M. Farquharson, P. O'Shea, and G. Ledwich, "A computationally efficient technique for estimating the parameters of polynomial-phase signals from noisy observations," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3337–3342, Aug. 2005.

[7] J. Kitchen, "A method for estimating the coefficients of a polynomial phase signal," *Signal Processing*, vol. 37, no. 1, pp. 463–470, Feb 1994.

[8] S. Golden and B. Friedlander, "Maximum likelihood estimation, analysis, and applications of exponential polynomial signals," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1493–1501, Jun 1999.

[9] S. Peleg and B. Porat, "The Cramer-Rao lower bound for signals with constant amplitude and polynomial phase," *IEEE Trans. Signal Process.*, vol. 39, no. 3, pp. 749–752, Mar. 1991.

[10] D. C. Rife and R. R. Boorstyn, "Single-tone parameter estimation from discrete-time observations," *IEEE Trans. Inform. Theory*, vol. 20, pp. 591–598, Sep. 1974.

[11] B. G. Quinn and E. J. Hannan, *The Estimation and Tracking of Frequency*, Cambridge University Press, New York, 2001.

[12] S. Golden and B. Friedlander, "A modification of the discrete polynomial transform," *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1452–1455, May 1998.

[13] S. A. Tretter, "Estimating the frequency of a noisy sinusoid by linear regression," *IEEE Trans. Inform. Theory*, vol. 31, no. 6, pp. 832–835, November 1985.

[14] S. M. Kay, "A fast and accurate single frequency estimator," *Proc. Internat. Conf. Acoust. Speech Signal Process.*, vol. 37, pp. 1987–1990, 1989.

[15] I. V. L. Clarkson, "Frequency estimation, phase unwrapping and the nearest lattice point problem," *Proc. Internat. Conf. Acoust. Speech Signal Process.*, vol. 3, pp. 1609–1612, Mar. 1999.

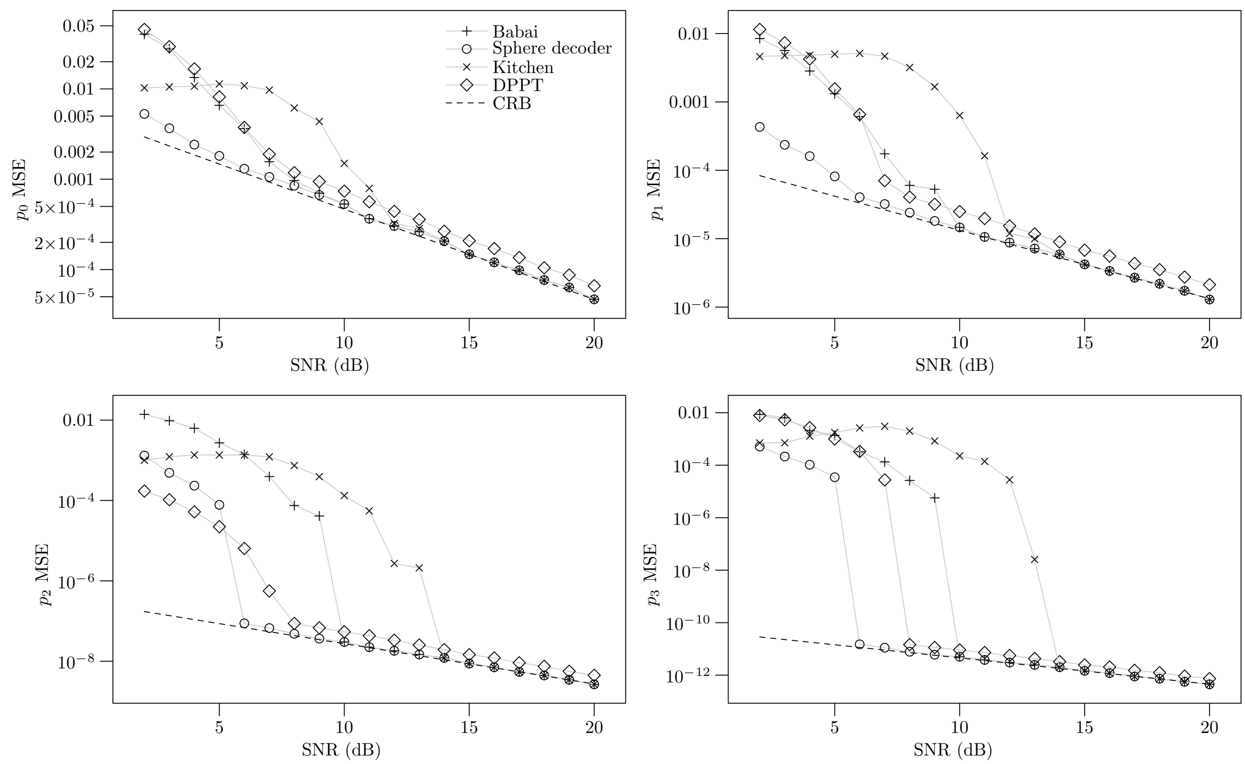

Fig. 1. MSE in each of the four polynomial parameters with $N = 50$ and $\mathbf{p} = [0.1, 0.05, 0.01, 0.0005]$.

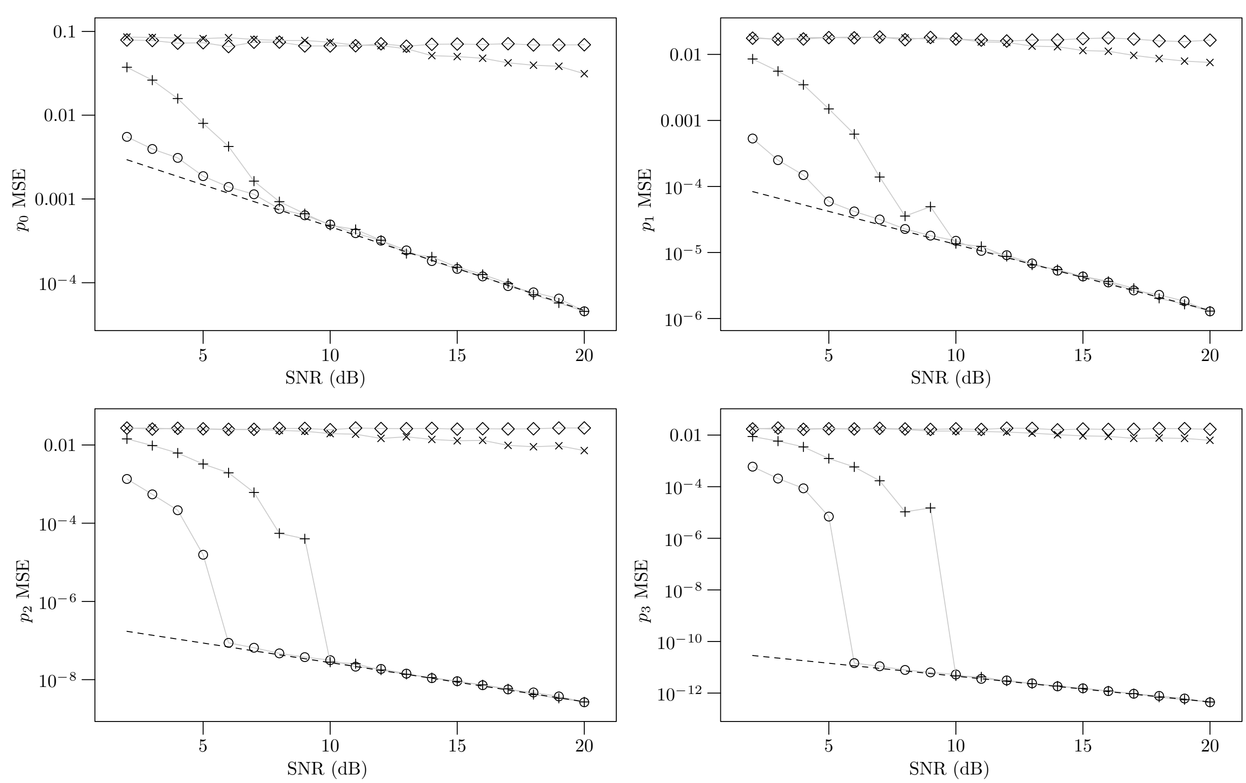

Fig. 2. MSE in each of the four polynomial parameters with $N = 50$ and $\mathbf{p}$ uniformly distributed in R.

[16] B.J. Slocumb and J. Kitchen, "A polynomial phase parameter estimation phase unwrapping algorithm," *Proc. Internat. Conf. Acoust. Speech Signal Process.*, vol. 4, pp. 129–132, Apr 1994.
[17] R. G. McKilliam and I. V. L. Clarkson, "Identifiability and aliasing in polynomial-phase signals," *IEEE Trans. Signal Process.*, vol. 57, no. 11, pp. 4554–4557, Nov. 2009.
[18] B. G. Quinn, "Estimating the mode of a phase distribution," *Asilomar Conference on Signals, Systems and Computers*, pp. 587–591, Nov 2007.
[19] R. G. McKilliam, Barry G. Quinn, I. V. L. Clarkson, and Bill Moran, "Frequency estimation by phase unwrapping," accepted for *IEEE Trans. Signal Process.*, 2009.
[20] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices," *IEEE Trans. Inform. Theory*, vol. 48, no. 8, pp. 2201–2214, Aug. 2002.
[21] M. Pohst, "On the computation of lattice vectors of minimal length, successive minima and reduced bases with applications," *SIGSAM Bull.*, vol. 15, no. 1, pp. 37–44, 1981.
[22] E. Viterbo and J. Boutros, "A universal lattice code decoder for fading channels," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1639–1642, Jul. 1999.
[23] L. Babai, "On Lovász lattice reduction and the nearest lattice point problem," *Combinatorica*, vol. 6, pp. 1–13, 1986.
[24] K. V. Mardia and P. Jupp, *Directional Statistics*, Wiley, 2nd edition, 2000.
[25] N. I. Fisher, *Statistical analysis of cirular data*, Cambridge University Press, 1993.
[26] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Springer, New York, 3rd edition, 1998.
[27] D. Micciancio, "The hardness of the closest vector problem with preprocessing," *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp. 1212–1215, 2001.
[28] M. Ajtai, "The shortest vector problem in L_2 is NP-hard for randomized reductions," *in Proc. 30th ACM Symposium on Theory of Computing*, pp. 10–19, May 1998.
[29] I. V. L. Clarkson, "An algorithm to compute a nearest point in the lattice A_n^*," in *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, Marc Fossorier, Hideki Imai, Shu Lin, and Alain Poli, Eds., vol. 1719 of *Lecture Notes in Computer Science*, pp. 104–120. Springer, 1999.
[30] R. G. McKilliam, I. V. L. Clarkson, and B. G. Quinn, "An algorithm to compute the nearest point in the lattice A_n^*," *IEEE Trans. Inform. Theory*, vol. 54, no. 9, pp. 4378–4381, Sep. 2008.
[31] R. G. McKilliam, W. D. Smith, and I. V. L. Clarkson, "Linear-time nearest point algorithms for Coxeter lattices," *IEEE Trans. Inform. Theory*, vol. 56, no. 3, pp. 1015–1022, Mar. 2010.
[32] R. G. McKilliam, I. V. L. Clarkson, W. D. Smith, and B. G. Quinn, "A linear-time nearest point algorithm for the lattice A_n^*," *International Symposium on Information Theory and its Applications*, Dec. 2008.
[33] A. K. Lenstra, H. W. Lenstra, and L. Lovász, "Factoring polynomials with rational coefficients," *Math. Ann.*, vol. 261, pp. 515–534, 1982.