

Identifiability and aliasing in polynomial-phase signals

Robby G. McKilliam and I. Vaughan L. Clarkson

Abstract—Polynomial-phase signals have numerous applications including radar, sonar, geophysics, and radio communication. Many techniques for estimating the parameters of polynomial-phase signals have been described in the literature. Despite the significant interest, aliasing of polynomial-phase parameters has not been fully clarified. We address the problem of identifiability and aliasing in polynomial-phase signals. We fully describe the region in which aliasing does not occur for polynomial-phase signals of any order. We call this the *identifiable region*. We find that this region is the *Voronoi region* of a lattice generated by the coefficients of a set of polynomials known as the *integer-valued polynomials*. We show how aliasing can be resolved by solving the *nearest lattice point problem*. We discuss some of the consequences of these results on a popular estimator for polynomial-phase signals that is based on the discrete polynomial phase transform (DPPT). It is shown that the range of parameters suitable for the DPPT estimator is very small compared to the identifiable region.

Index Terms—Polynomial-phase signals, lattice theory, antialiasing, parameter estimation, chirp signals, polynomials, radar signal processing, Doppler measurements, closest point search, lattice decoding, Voronoi diagram.

I. INTRODUCTION

A uniformly sampled polynomial-phase signal of order m has the model [1–4],

$$z[n] = A[n] \exp \left(2\pi j \sum_{k=0}^m p_k (\Delta n)^k \right) \quad (1)$$

where $n \in \mathbb{Z}$, $A[n]$ is the signal amplitude at the n th sample, the p_k are the parameters, $j = \sqrt{-1}$ and Δ is the interval between consecutive samples. We will assume, without loss of generality, that $\Delta = 1$. We will often write the parameters, p_k , as a column vector \mathbf{p} .

Polynomial-phase signals have numerous applications including radar, sonar, geophysics, and radio communication [5]. The signals are also used to describe the sounds emitted by bats for echolocation [1]. Of significant practical importance is the estimation of the parameters \mathbf{p} from the signal $z[n] + s[n]$ where $s[n]$ is a noise process. Many estimators for polynomial-phase parameters have been studied and implemented [1–3, 6–12].

Despite the significant interest in polynomial-phase signals, aliasing of polynomial-phase parameters has not been fully clarified [5]. Inherent ambiguities exist in (1). For example, consider the case when $m = 0$. The model becomes

$$z[n] = A[n] \exp(2\pi j p_0 n)$$

where $p_0 \in \mathbb{R}$. Then $z[n]$ is identical for $p_0 + c$ for any $c \in \mathbb{Z}$. In order to avoid these ambiguities we must restrict p_0 to some interval of length 1. A natural choice is $[-1/2, 1/2)$. We call this the *identifiable region*. For the case when $m = 1$ we find that the identifiable region is $(p_0, p_1) \in [-1/2, 1/2)^2$ which corresponds with the Nyquist criterion.

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Robby McKilliam is partly supported by a scholarship from the Wireless Technologies Laboratory, CSIRO ICT Centre, Sydney, Australia

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For larger m the identifiable region becomes more complicated. The identifiable regions for $m \leq 2$ have been found by Ångeby [5] but a general description does not appear to exist in the literature. In this paper, we fully describe the identifiable region for all $m \geq 0$. The region is shown to be the *Voronoi region* of a lattice. The lattice points are given by the coefficients of a set of polynomials known as the *integer-valued polynomials* [13].

Some work on resolving aliasing in polynomial-phase signals has been conducted by Ångeby [5]. Ångeby suggests that aliasing can be avoided by nonuniform sampling. Specifically, the interval between some of the samples should be irrational. A similar approach was suggested by Legg and Gray [14]. However, they used randomly generated sampling times.

Although these approaches avoid aliasing in the noise free case, it can be shown that they lead to large estimation error when noise is added. In effect, the previously ambiguous parameters are now “close” to each other. An estimator is then likely to incorrectly choose one of the *almost* ambiguous parameters. This is a result of Kronecker’s approximation theorem [15, pp. 148–155]. We will briefly describe this problem. Let \mathbf{t} be a vector of (potentially non-uniform) sampling times. Let \mathbf{p} be a set of polynomial phase parameters and define

$$z(\mathbf{p})[n] = \exp \left(2\pi j \sum_{k=0}^m p_k t[n]^k \right)$$

Using Kronecker’s approximation theorem it can be shown that for any $\epsilon > 0$ there exists a $\mathbf{p}^* \neq \mathbf{p}$ such that $\|\mathbf{p}^* - \mathbf{p}\| > \delta$ for any $\delta > 0$ and

$$|z(\mathbf{p})[n] - z(\mathbf{p}^*)[n]|^2 < \epsilon \quad (2)$$

for all $n = 1, 2, \dots, N$ where $|\cdot|$ is the complex magnitude. In fact there are an infinite number of such \mathbf{p}^* for arbitrarily large δ . Consider the noisy signal $z(\mathbf{p})[n] + s[n]$. The least squares estimator of \mathbf{p} is

$$\arg \min_{\mathbf{p}^* \in \mathbb{R}^{m+1}} \sum_{n=1}^N |z(\mathbf{p})[n] + s[n] - z(\mathbf{p}^*)[n]|^2.$$

Considering (2) we see that this minimization may result in one of the infinite number of *almost* ambiguous \mathbf{p}^* rather than the true value \mathbf{p} . We have fixed $A[n] = 1$ in this example, but a similar result can be shown for arbitrary $A[n]$.

Attempts at avoiding aliasing by modifying the sampling interval were probably motivated by the difficulty of describing the identifiable region. We solve this problem in this paper. The paper is organized as follows. In Section II we introduce some basic concepts in lattice theory. This includes the Voronoi region and the nearest lattice point problem. In Section III we describe the identifiable region. We show that the identifiable region is the Voronoi region of a lattice. In Section IV we describe how to resolve aliasing in polynomial-phase parameters. We also show how to calculate the square error between two parameters and how to generate parameters that are uniformly distributed in the identifiable region. Finally, we discuss the consequences of these results on a popular existing estimator based on the discrete polynomial-phase transform (DPPT) [1, 3]. It is shown that the range of parameters suitable for the DPPT estimator is very small compared to the identifiable region.

II. LATTICE THEORY

A lattice, L , is a set of points in \mathbb{R}^n such that

$$L = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{B}\mathbf{w}, \mathbf{w} \in \mathbb{Z}^n\}$$

where \mathbf{B} is called the *generator* or *basis matrix* [16]. The *Voronoi region* or *nearest-neighbor region*, $\text{Vor}(L)$, for a lattice L is the subset of \mathbb{R}^n such that, with respect to a given norm, all points in $\text{Vor}(L)$ are nearer to the origin than to any other point in L . The Voronoi region is an n dimensional polytope [16]. Given some lattice point $\mathbf{x} \in L$ we will write $\text{Vor}(L) + \mathbf{x}$ to denote the Voronoi region centered around the lattice point \mathbf{x} . It follows that $\text{Vor}(L) + \mathbf{x}$ is the subset of \mathbb{R}^n that is nearer to \mathbf{x} than any other lattice point in L . Figure 1 is an example of a lattice and its Voronoi region in \mathbb{R}^2 . This lattice has basis matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1/5 \\ 1/5 & 1 \end{pmatrix}$$

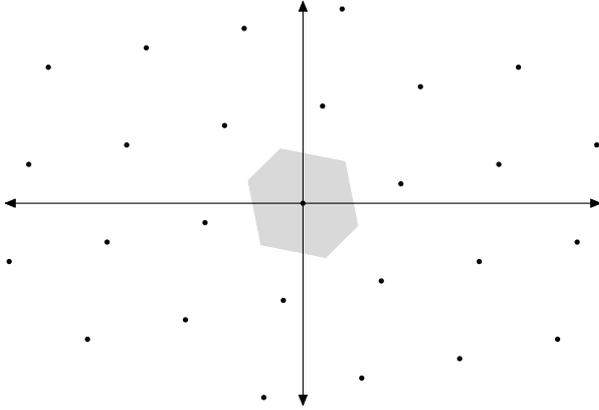


Fig. 1. A lattice in \mathbb{R}^2 . The shaded region is the Voronoi region.

A fundamental problem in lattice theory is the *nearest lattice point problem*. The nearest lattice point problem is, given $\mathbf{y} \in \mathbb{R}^n$ and some lattice L whose lattice points lie in \mathbb{R}^n , to find a lattice point $\mathbf{x} \in L$ such that, with respect to a given norm, the distance between \mathbf{y} and \mathbf{x} is minimized. We will use the notation $\text{NearestPt}(\mathbf{y}, L)$ to denote the nearest point in L to \mathbf{y} . Clearly¹

$$\mathbf{x} = \text{NearestPt}(\mathbf{y}, L) \Leftrightarrow \mathbf{y} \in \text{Vor}(L) + \mathbf{x}.$$

The nearest lattice point problem is known to be NP-hard under certain conditions when the lattice itself, or rather a basis thereof, is considered as an additional input parameter [17, 18]. Nevertheless, algorithms exist that can compute the nearest lattice point in reasonable time if the dimension is small [19–21]. One such algorithm introduced by Pohst [21] in 1981 was popularized in signal processing and communications fields by Viterbo and Boutros [20] and has since been called the *sphere decoder*.

III. THE IDENTIFIABLE REGION

In this section we show that the identifiable region of (1) is the Voronoi region of a lattice in \mathbb{R}^{m+1} . The lattice points are given by the coefficients of a set of polynomials called the integer-valued

¹There is a slight technical deficiency here. We actually require to define half of the faces of $\text{Vor}(L)$ to be closed and half to be open. Ties in $\text{NearestPt}(\mathbf{y}, L)$ can then be broken accordingly.

polynomials. Assume that the parameters \mathbf{p} and $\mathbf{p} + \mathbf{v}$ are ambiguous. Then

$$\begin{aligned} & A[n] \exp\left(2\pi j \sum_{k=0}^m p_k n^k\right) \\ &= A[n] \exp\left(2\pi j \sum_{k=0}^m (p_k + v_k) n^k\right) \\ &= A[n] \exp\left(2\pi j \sum_{k=0}^m p_k n^k\right) \exp\left(2\pi j \sum_{k=0}^m v_k n^k\right) \end{aligned}$$

and therefore

$$\exp\left(2\pi j \sum_{k=0}^m v_k n^k\right) = 1$$

which occurs if and only if

$$\sum_{k=0}^m v_k n^k \in \mathbb{Z}.$$

for all $n \in \mathbb{Z}$. We consider the set of all such \mathbf{v}

$$L = \left\{ \mathbf{v} \in \mathbb{R}^{m+1} \mid \sum_{k=0}^m v_k n^k \in \mathbb{Z}, n \in \mathbb{Z} \right\}.$$

Let $\text{Int}_m[\mathbb{Z}]$ denote the set of polynomials of order m that are integer valued when evaluated at integers. Given a polynomial P of order at most m , define $\text{Coef}(P)$ to be the column vector of length $m+1$ containing the coefficients of P . If P is of order $k < m$ then the last $m-k$ elements of $\text{Coef}(P)$ are zero. Then

$$L = \{\mathbf{v} = \text{Coef}(P) \mid P \in \text{Int}_m[\mathbb{Z}]\}. \quad (3)$$

Define the polynomials

$$P_k[n] = \frac{n(n+1)(n+2)\dots(n+k-1)}{k!}$$

and define $P_0[n] = 1$. We will often drop the $[n]$ and write P_k as P_k . These polynomials are known as the *integer-valued polynomials* and have been extensively studied [13]. The following lemma and proof is adapted from Cahen and Chabert [13, p. 2]

Lemma 1. *The P_k , for $k = 0, 1, \dots, m$, are an integer basis for $\text{Int}_m[\mathbb{Z}]$. That is, every element of $\text{Int}_m[\mathbb{Z}]$ can be uniquely written as*

$$c_0 P_0 + c_1 P_1 + \dots + c_m P_m \quad (4)$$

where the $c_i \in \mathbb{Z}$.

Proof: Note that $n(n+1)(n+2)\dots(n+k-1)$ is divisible by all integers $1, 2, \dots, k$ and so P_k takes integer values for all $n \in \mathbb{Z}$. Then any polynomial generated as in (4) is an element in $\text{Int}_m[\mathbb{Z}]$. The proof proceeds by induction. Consider any polynomial $f \in \text{Int}_m[\mathbb{Z}]$. Let $d < n$ and assume that $c_i \in \mathbb{Z}$ for all $i \leq d$. Let g be the polynomial

$$g = f - \sum_{k=0}^d c_k P_k$$

and note that $g \in \text{Int}_m[\mathbb{Z}]$. Then

$$g = c_{d+1} P_{d+1} + \dots + c_m P_m. \quad (5)$$

Now $P_{d+1}[-d-1] = \pm 1$ and $P_k[-d-1] = 0$ for all $k > d+1$. Then $g[-d-1] = c_{d+1} P_{d+1}[-d-1]$ and therefore $c_{d+1} = \pm g[-d-1] \in \mathbb{Z}$. The proof follows by induction because $f[0] = c_0 \in \mathbb{Z}$. ■

Theorem 1. *Construct the $(m+1) \times (m+1)$ matrix with columns $\text{Coef}(P_k)$ for $k = 0, 1, \dots, m$, i.e.*

$$\mathbf{B} = [\text{Coef}(P_0) \text{Coef}(P_1) \dots \text{Coef}(P_m)].$$

Then

$$L = \{\mathbf{v} = \mathbf{B}\mathbf{w} \mid \mathbf{w} \in \mathbb{Z}^{m+1}\}$$

and it is clear that L is a lattice in \mathbb{R}^{m+1} with basis matrix \mathbf{B} .

Proof: The proof follows directly from consideration of (3) and Lemma 1. ■

The lattice points in L are the parameters that are ambiguous with the origin ($\mathbf{p} = \mathbf{0}$). The identifiable region is the Voronoi region of L^2 . As an example, consider when $m = 3$,

$$\begin{aligned} P_0 &= 1 \\ P_1 &= n \\ P_2 &= \frac{n^2}{2} + \frac{n}{2} \\ P_3 &= \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}. \end{aligned}$$

The basis matrix for L is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}$$

and the identifiable region is $\text{Vor}(L)$.

IV. RESOLVING ALIASING, COMPUTING ERROR AND GENERATING PARAMETERS

In this section we show how, given some polynomial-phase parameter not necessarily in the identifiable region, we can find the equivalent parameter in the identifiable region. For evaluating the performance of estimators it is usually necessary to calculate the square error between the true and estimated parameters. We show how the square error can be calculated unambiguously. Finally we show how to generate polynomial-phase parameters that are uniformly distributed in the identifiable region. This is useful if we wish to evaluate the performance of an estimator over the entire identifiable region. These procedures are facilitated by solving the nearest point problem for the lattice L . In practice m is typically small and the nearest point can be efficiently computed by sphere decoding [19–21].

Given some parameter \mathbf{x} the equivalent parameter within the identifiable region is

$$\mathbf{p} = \mathbf{x} - \text{NearestPt}(\mathbf{x}, L).$$

When estimating the parameters of polynomial-phase signals we usually have some true parameters \mathbf{p} and the estimated parameters $\hat{\mathbf{p}}$. We often wish to compute the square error between the true and estimated parameters. Some difficulties arise due to aliasing. For example, consider when $m = 0$. It may be that the true parameter $p_0 = 0.49$ and the estimated parameter $\hat{p}_0 = -0.49$. Naively we might compute the square error as $(p_0 - \hat{p}_0)^2 = 0.98^2$. Intuitively this is wrong because πp_0 and $\pi \hat{p}_0$ are phases that are close together, but lie on either side of the branch on the unit circle. We can correctly compute the square error as

$$(\hat{p}_0 - p_0 - \lfloor \hat{p}_0 - p_0 \rfloor)^2 = 0.02^2$$

²Note we may choose other regions that tessellate the space. For example, a fundamental parallelepiped of the lattice or the $m+1$ dimensional rectangular prism $\prod_{k=0}^m [-0.5/k!, 0.5/k!]$. However, the Voronoi region is the most natural choice.

where $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer³. Analogously, to compute the square error for any $m \geq 0$ we first compute

$$\mathbf{g} = \hat{\mathbf{p}} - \mathbf{p} - \text{NearestPt}(\hat{\mathbf{p}} - \mathbf{p}, L).$$

where $\text{NearestPt}(\cdot)$ is computed with respect to the 2-norm. The square error of the k th parameter is then g_k^2 .

A parameter uniformly distributed in the identifiable region can be generated as

$$\mathbf{p} = \text{NearestPt}(\mathbf{B}\mathbf{u}, L)$$

where \mathbf{B} is the basis matrix for L and \mathbf{u} is a vector whose elements are independent and uniformly distributed on $[0, 1)$.

V. CONSEQUENCES FOR SOME EXISTING ESTIMATORS

A popular estimator for polynomial-phase parameters is based on the discrete polynomial-phase transform (DPPT) first introduced by Peleg and Porat [1]. The transform enables each parameter to be estimated iteratively using the Fast Fourier transform in a way analogous to the frequency estimator of Rife and Boorstyn [22, 23]. The DPPT estimator is characterized by good statistical performance and computational efficiency. Variants of the DPPT have been suggested by numerous authors. These include Golden and Friedlander [24] and O’Shea [8]. These variants typically make trade offs between computational efficiency and statistical accuracy.

One property of the DPPT estimator is that it only functions when the parameters satisfy

$$2|p_k| \leq \frac{1}{k! \tau^{k-1}}$$

for $k \geq 2$ and $|p_k| \leq 0.5$ for $k = 0, 1$ and where τ is a ‘lag’ parameter that is free to be selected. Peleg and Porat suggest using $\tau = N/m$ when $m < 4$ and $\tau = N/(m+2)$ for $m \geq 4$ where N is the number of samples of data available. The functional region of the DPPT estimator is then an $m+1$ dimensional rectangular prism of volume

$$V_{DPPT} = \sqrt{\tau^{m(1-m)}} \prod_{k=0}^m \frac{1}{k!} \quad (6)$$

The volume of the identifiable region is the volume of $\text{Vor}(L)$ which is given by

$$V_L = \sqrt{\det(\mathbf{B}'\mathbf{B})} = \prod_{k=0}^m \frac{1}{k!}$$

where $\det(\cdot)$ indicates the matrix determinant [16]. Note that $\text{Vor}(L)$ depends only on m and not N and so the aliasing of polynomial phase signals is not dependent on N . By contrast the functional region of the DPPT estimator shrinks with increasing N because the parameter τ must be chosen to increase with N in order for the DPPT estimator to provide good estimation. Figure 2 shows the ratio V_L/V_{DPPT} as N increases from 10 to 100 and when $m = 3$. Clearly $V_{DPPT} \ll V_L$ for large N and therefore the range of parameters suitable for the DPPT estimator is only a small fraction of what is theoretically possible.

VI. CONCLUSION

Polynomial-phase signals have attracted significant interest due to their applicability to radar, sonar, geophysics, and radio communication. Despite this interest the aliasing of polynomial-phase parameters had never been fully described. In this paper we describe the identifiable region for the parameters. The region is found to be the Voronoi region of a lattice with lattice points described

³The direction of rounding for half-integers is not important. However, the authors have chosen to round up half-integers here.

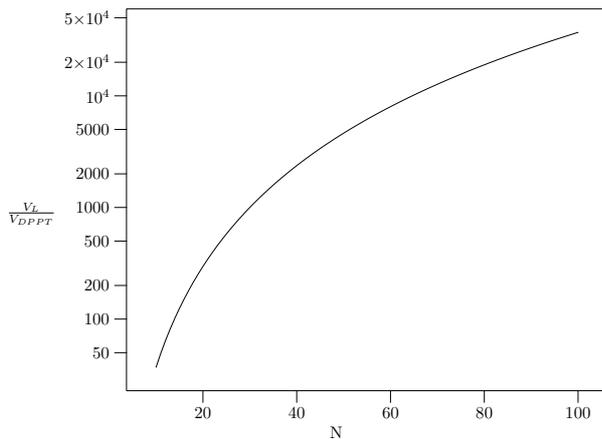


Fig. 2. V_L/V_{DPPT} as N increase when $m = 3$.

by the coefficients of the integer valued polynomials. By solving the nearest lattice point problem we show how to resolve aliased parameters, compute square error and generate parameters uniformly in the identifiable region. Finally, we discuss the consequences of these results on the popular DPPT estimator for polynomial-phase signals [1, 3]. It is shown that the range of parameters suitable for the DPPT estimator is very small compared to the identifiable region.

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