Testable linear shift-invariant systems

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Contents

Chapter 1

Signals and systems

It is assumed that the reader is familiar with the concept of a function! That is, a map from the elements in a set X to the elements in another set Y. Consider sets

$$X = \left\{ \begin{array}{c} \text{Mario} \\ \text{Link} \\ \text{Ness} \end{array} \right\} \qquad Y = \left\{ \begin{array}{c} \text{Freeman} \\ \text{Ryu} \\ \text{Sephiroth} \\ \text{Conker} \\ \text{Ness} \end{array} \right\}.$$

An example of a function from X to Y is

$$f(x) = \begin{cases} \text{Conker} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \\ \text{Sephiroth} & x = \text{Ness}. \end{cases}$$

The function f maps Mario to Conker, Link to Sephiroth, and Ness to Sephiroth. The value of f for input x is denoted by $f(x)$ and so, for example, $f(\text{Mario}) = \text{Conker}$ and $f(\text{Link}) = \text{Sephiroth}$. The set X is called the **domain** of the function f and the set of values that the functions takes, that is, the set $\{f(x), x \in X\}$, is called the **range**. In the above example the range is the set $\{\text{Conker}, \text{Sephiroth}\}$. Observe that the range is a subset of Y. The set of all functions mapping X to Y is denoted by $X \to Y$ and so $f \in X \to Y$ in the example above.

A **signal** is a function that maps a real number to a complex number, that is, a function from the set $\mathbb{R} \to \mathbb{C}$. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent signals with $t \in \mathbb{R}$. The real part of these signals is plotted in Figure 1.1. Many physical phenomena, such as sound, light, weather,

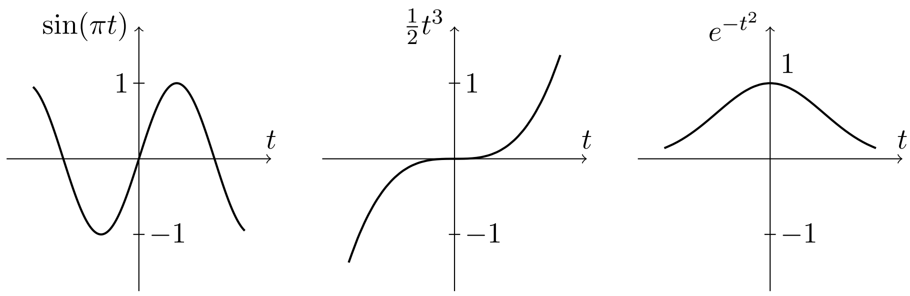

Figure 1.1: Plots of three signals.

and motion, can be modelled using signals. In this text we primarily focus on examples from electrical and mechanical engineering where signals are use to model changes in quantities such as voltage, current, position, angle, force, and torque, over time. In these examples, the independent variable t represents "time". However, there is no fundamental reason for this and the techniques developed here can be applied equally well when t represents a quantity other than time. An example where this occurs is image processing.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a positive real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

The smallest such positive T it is called the **fundamental period**. For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is **right sided** if there exists $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.1.1}$$

is right-sided. Its horizontal reflection $u(-t)$ is left sided (Figure 1.2). A signal x is said to be **finite** or to have **finite support** if it is both left and

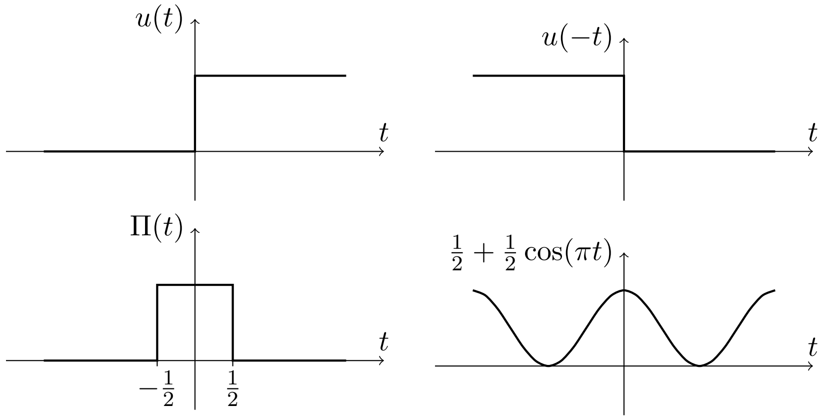

Figure 1.2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite rectangular pulse $\Pi(t)$ and the signal $\frac{1}{2} + \frac{1}{2}\cos(x)$ that is not finite.

right sided, that is, if there exists $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. The signals $\sin(\pi t)$ and e^{-t^2} do not have finite support, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.2}$$

does.

A signal x is **even** (or **symmetric**) if

$$x(t) = x(-t) \qquad \text{for all } t \in \mathbb{R}$$

and **odd** (or **antisymmetric**) if

$$x(t) = -x(-t) \qquad \text{for all } t \in \mathbb{R}.$$

For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are odd and e^{-t^2} is even. A signal x is **conjugate symmetric** if

$$x(t) = x(-t)^* \qquad \text{for all } t \in \mathbb{R}$$

and **conjugate antisymmetric** if

$$x(t) = -x(-t)^* \qquad \text{for all } t \in \mathbb{R},$$

where $*$ denotes the complex conjugate of a complex number. Equivalently, x is conjugate symmetric if its real part $\operatorname{Re} x$ is an even signal and its imaginary part $\operatorname{Im} x$ is an odd signal, and x is conjugate antisymmetric if its real part is odd and its imaginary part is even. For example, the signal $e^{-t^2} + j\sin(\pi t)$ where $j = \sqrt{-1}$ is conjugate symmetric and the signal $\frac{1}{2}t^3 + je^{-t^2}$ is conjugate antisymmetric.

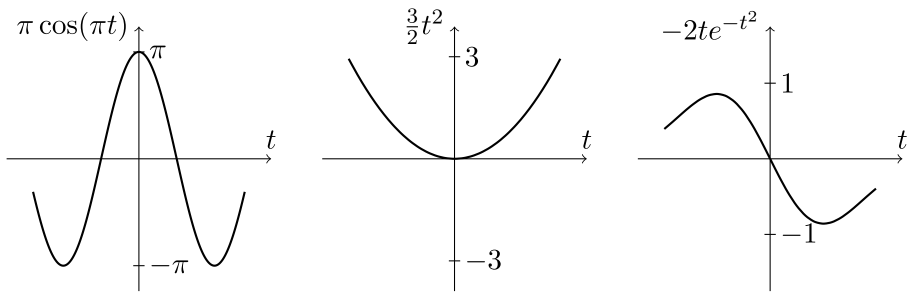

Figure 1.3: Derivatives of the signals $\sin(\pi t)$, $\frac{1}{2}t^3$, e^{-t^2} from Figure 1.1.

A signal x is **continuous at** $t \in \mathbb{R}$ if

$$\lim_{h\to 0} x(t+h) = \lim_{h\to 0} x(t-h)$$

and x is said to be **continuous** if it is continuous at all $t \in \mathbb{R}$. The signals $\sin(\pi t)$, $\frac{1}{2}t^3$, and e^{-t^2} are continuous, but the step function u is not continuous at zero because

$$\lim_{h\to 0} u(h) = 1 \neq 0 = \lim_{h\to 0} u(-h).$$

The set of continuous signals is typically denoted by $C^0(\mathbb{R})$ or just C^0. A signal x is **continuously differentiable** or just **differentiable** if

$$\lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h\to 0} \frac{x(t) - x(t-h)}{h} \qquad \text{for all } t \in \mathbb{R}.$$

Considered as a function of t this limit is called the **derivative** of x at t and is typically denoted by $\frac{d}{dt}x(t)$. For example, the signals $\sin(\pi t)$, $\frac{1}{2}t^3$, e^{-t^2}, and t^2 are differentiable with derivatives

$$\pi\cos(\pi t), \qquad \tfrac{3}{2}t^2, \qquad -2te^{-t^2}, \qquad 2t,$$

but the step function u and the rectangular pulse Π are not differentiable (Exercise 1.7). The set of differentiable signals is denoted by C^1 or $C^1(\mathbb{R})$. A signal is **k-times differentiable** if its $k-1$ th derivative is differentiable. The set of k-times differentiable signals is typically denoted by C^k or $C^k(\mathbb{R})$.

A signal x is **locally integrable** if

$$\int_a^b |x(t)|\, dt < \infty$$

for all finite constants a and b, where $< \infty$ means that the integral evaluates finite complex number. The signals $\sin(\pi t)$, $\frac{1}{2}t^3$, and e^{-t^2} are all locally integrable. An example of a signal that is not locally integrable is $x(t) = \frac{1}{t}$

(Exercise 1.3). The set of locally integrable signals is typically denoted by L_{loc} or $L_{\text{loc}}(\mathbb{R})$.

A signal x is **absolutely integrable** or **Lebesgue integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt < \infty. \tag{1.1.3}$$

Here we introduce the notation $\|x\|_1$ called the L^1**-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.1.4}$$

It is common to denote the set of absolutely integrable signals by L^1 or $L^1(\mathbb{R})$. So, $e^{-t^2} \in L^1$ and $\frac{1}{2}t^3 \notin L^1$. A signal x is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty.$$

The real number $\|x\|_2$ is called the L^2**-norm** of x. Square integrable signals are also called **energy signals** and the squared L^2-norm $\|x\|_2^2$ is called the **energy** of x. For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is. It has energy $\|e^{-t^2}\|_2^2 = \sqrt{\pi/2}$ (Exercise 1.6). The set of square integrable signals is denoted by L^2 or $L^2(\mathbb{R})$.

We write $x = y$ to indicate that two signals x and y are **equal pointwise**, that is, $x(t) = y(t)$ for all $t \in \mathbb{R}$. This definition of equality is often stronger than we desire. For example, the step function u and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at $t = 0$ since $u(0) = 1$ and $z(0) = 0$. It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals x and y are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)|\, dt = 0$$

for all finite constants a and b. So, in the previous example, while $u \neq z$ pointwise we do have $u = z$ almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \;\text{ a.e.} \qquad \text{or} \qquad x(t) = y(t) \;\text{ a.e.}$$

to indicate that the signals x and y are equal almost everywhere.

1.2 Spaces of signals

We will regularly be interested in subsets of the set of all signals $\mathbb{R} \to \mathbb{C}$. Two important families of subsets are the **linear spaces** and the **shift-invariant spaces**.

Let x and y be signals. We denote by $x+y$ the signal that takes the value $x(t)+y(t)$ for each $t \in \mathbb{R}$, that is, the signal that results from adding x and y. For $a \in \mathbb{C}$ we denote by ax the signal that takes the value $ax(t)$ for each $t \in \mathbb{R}$, that is, the signal that results from multiplying x by a (Figure 1.4). For signals x and y and complex numbers a and b the signal

$$ax + by$$

is called a **linear combination** of x and y.

Let $X \subseteq \mathbb{R} \to \mathbb{C}$ be a set of signals. The set X is a **linear space** (or **vector space**) if for all signals x and y from X and all complex numbers a and b the linear combination $ax + by$ is also in X. The set of all signals $\mathbb{R} \to \mathbb{C}$ is a linear space. Another example is the set of differentiable signals, because, if x and y are differentiable, then the linear combination $ax + by$ is differentiable. The derivative is $aDx + bDy$. The set of even signals is another example of a linear space because if x and y are even then

$$ax(t) + by(t) = ax(-t) + by(-t)$$

and so the linear combination $ax + by$ is even. The set of locally integrable signals L_{loc}, the set of absolutely integrable signals L^1, and the set of square integrable signals L^2 are linear spaces (Exercise 1.9). The set of periodic signals is not a linear space (Exercise 1.10).

For a real number τ the signal $x(t-\tau)$ is called a **time-shift** or **shift** or sometimes **translation** of the signal $x(t)$. Figure 1.5 depicts the shift $x(t-\tau)$ for different values of τ in the case that $x(t) = e^{-t^2}$. A set of signals $X \subseteq \mathbb{R} \to \mathbb{C}$ is a **shift-invariant space** if for all $x \in X$ and all $\tau \in \mathbb{R}$ the shift $x(t-\tau)$ is also in X. Examples of shift-invariant spaces are the set of differentiable signals, the set of periodic signals (Exercise 1.10), and L_{loc}, L^1, and L^2 (1.9). The set of even signals and the set of odd signals are not shift-invariant spaces.

Linear spaces and shift-invariant spaces of signals will act as domains for the next type of function that we consider called **systems**.

1.3 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example,

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

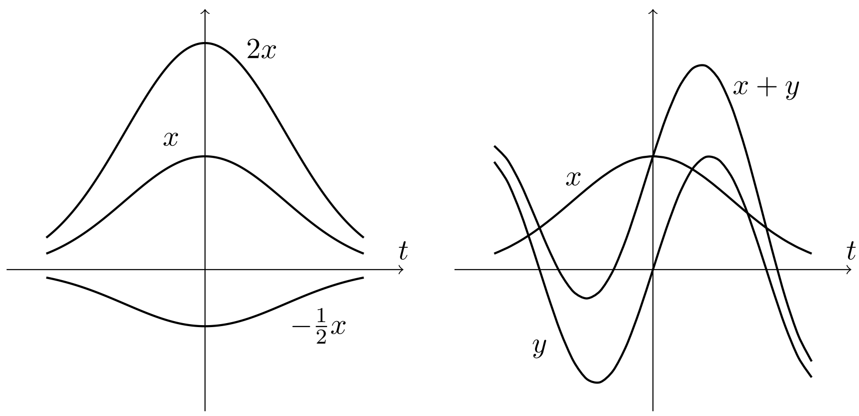

Figure 1.4: The signal $x(t) = e^{-t^2}$ and the signals $2x$ and $-\frac{1}{2}x$ (left). The signals $x(t) = e^{-t^2}$ and $y(t) = \sin(\pi t)$ and the signal $x + y$.

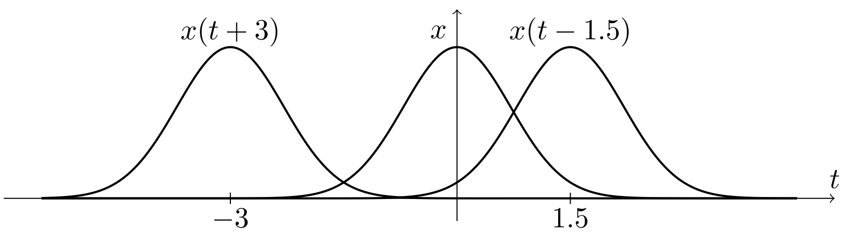

Figure 1.5: The signal $x(t) = e^{-t^2}$ and shift $x(t - \tau) = e^{-(t-\tau)^2}$ for $\tau = 1.5$ and -3.

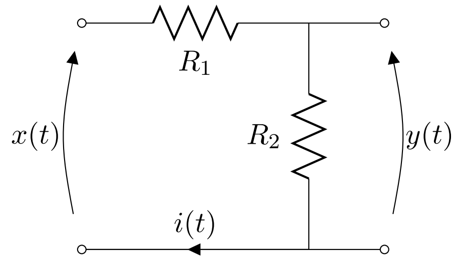

Figure 1.6: A **voltage divider** circuit.

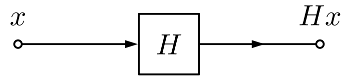

Figure 1.7: System block diagram with input signal x and output signal $H(x)$.

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 1.6 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \tag{1.3.1}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

Let $X \subseteq \mathbb{R} \to \mathbb{C}$ and $Y \subseteq \mathbb{R} \to \mathbb{C}$ be sets of signals. We denote systems with capital letters such as H and G. A system is a function $H \in X \to Y$ that maps each signal from the domain X to a signal from Y. Given **input signal** $x \in X$ the **output signal** of the system is denoted by $H(x)$. The output signal is often called the **response** of system H to signal x. We will often drop the brackets and write simply Hx for the response of H to x[1]. The value of the output signal Hx at $t \in \mathbb{R}$ is denoted by $Hx(t)$ or $H(x)(t)$ or $H(x,t)$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram as in Figure 1.7. The electric circuit in Figure 1.6 corresponds with the system

$$Hx = \frac{R_2}{R_1 + R_2} x = y.$$

[1] In the literature it is customary to drop the brackets only when H is a **linear** system (Section 1.5). In this text the we occasionally drop the brackets even when H is not linear. Since we deal primarily with linear systems this faux pas will occur rarely.

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 1.6 on a breadboard with resistors $R_1 \approx 100\Omega$ and $R_2 \approx 470\Omega$ with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F = \frac{1}{P} = 44100\text{Hz}$ to generate samples

$$x(nP) \qquad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor R_2 is recorded (also using the soundcard) that returns a list of samples $y_1, \dots, y_L$ taken at rate F. The voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(Ft - \ell) \tag{1.3.2}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.3.3}$$

is the called the **sinc function** and is plotted in Figure 5.1. We will justify this reconstruction in Section 5.4. Simultaneously the (stereo) soundcard is used to record the input voltage x producing samples $x_1, \dots, x_L$ taken at rate F. An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - \ell). \tag{1.3.4}$$

In view of (1.3.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2}\tilde{x} = \frac{47}{57}\tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{47}{57}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 1.8. The hypothesised output signal $\frac{47}{57}\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

When specifying a system we are free to choose the domain X at our convenience. In cases such as the voltage divider it is reasonable to choose the domain $X = \mathbb{R} \to \mathbb{C}$, that is, the domain can contain *all* signals. However, this is not always convenient or possible. For example, the system

$$Hx(t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. To avoid this we might choose the domain as the set of signals $x(t)$ that are not zero for any $t \in \mathbb{R}$.

Another example is the system I_∞ defined by

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau, \tag{1.3.5}$$

called an **integrator**. The signal $x(t) = 1$ cannot be input to the integrator because the integral $\int_{-\infty}^{t} dt$ is not finite for any t. However, the integrator I_∞ can operate on absolutely integrable signals because, if x is absolutely integrable, then

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau \leq \int_{-\infty}^{t} |x(\tau)| \, d\tau < \int_{-\infty}^{\infty} |x(\tau)| \, d\tau = \|x\|_1 < \infty$$

for all $t \in \mathbb{R}$. We might then choose a domain for I_∞ as the set of absolutely integrable signals L^1. The integrator can also be applied to signals that are right sided and locally integrable because, for any right sided signal x there exists $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$ and so,

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau = \int_{T}^{t} x(\tau) d\tau < \infty$$

for all $t \in \mathbb{R}$ if x is locally integrable. So another possible domain for I_∞ is the set of right sided locally integrable signals. A final possible domain is the subset of locally integrable signals for which $\int_{-\infty}^{0} |x(t)| \, dt$ is finite. This last example will be the domain we usually choose for the integrator I_∞.

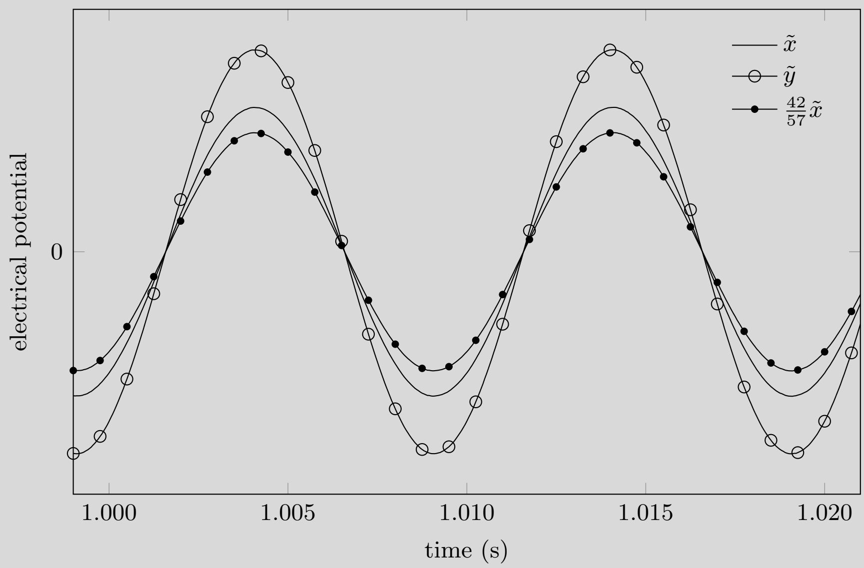

Figure 1.8: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{47}{57}\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 1.6. The hypothesised signal does not match $\tilde{y}$. One reason is that the model does not take account of the circuitry inside the soundcard.

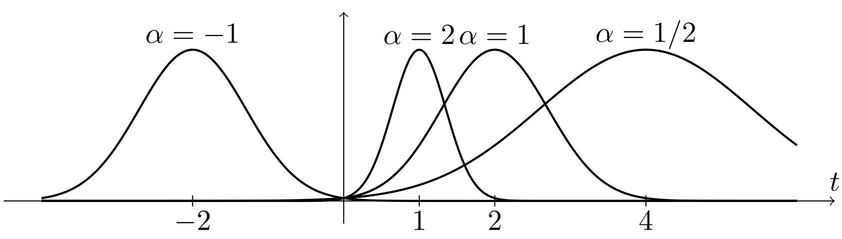

Figure 1.9: Time-scaler system $Hx(t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

1.4 Some important systems

The system

$$T_\tau x(t) = x(t - \tau)$$

is called a **time-shifter** or simply **shifter**. This system shifts the input signal along the t axis ("time" axis) by τ. When τ is positive T_τ delays the input signal by τ. The shifter will appear so regularly that we use the special notation T_τ to represent it. Figure 1.5 depicts the action of shifters $T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the shifter is the **identity system** $T_0 x = x$ that maps a signal to itself. Another important system is the **time-scaler** that has the form

$$Hx(t) = x(\alpha t), \qquad \alpha \in \mathbb{R}.$$

Figure 1.9 depicts the action of time-scalers with different values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the t axis. When $\alpha = 1$ the time-scaler is the identity system T_0. Both the shifter and time-scaler are well defined for all signals and so it is reasonable to choose their domains to be the entire set of signals $\mathbb{R} \to \mathbb{C}$. We always assume this is the case unless otherwise stated.

Another regularly encountered system is the **differentiator**

$$Dx(t) = \frac{d}{dt}x(t)$$

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k x(t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal. The differentiator is only defined for differentiable signals. A domain for D is the set of differentiable signals C^1 and a domain for D^k is the set of k-times differentiable signals C^k. Unless otherwise stated we will always assume the domain of D^k to be C^k.

Another important system is the **integrator**

$$I_a x(t) = \int_{-a}^{t} x(\tau) d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$. For example, the response of the integrator I_∞ to the signal $tu(t)$ where u is the step function (1.1.1) is

$$\int_{-\infty}^{t} \tau u(\tau) d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Observe that the integrator I_∞ cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ is not finite for any t. A domain for I_∞ cannot contain the signal $x(t) = t$. Unless otherwise stated we will assume the domain of I_∞ to be the subset of locally integrable signals L_{loc} for which $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ (Exercise 1.15). For finite $a < \infty$ we will assume, unless otherwise stated, that the domain of I_a is the set of locally integrable signals L_{loc}.

1.5 Properties of systems

In this section we define a number of important properties that systems can possess. In what follows $X \subseteq \mathbb{R} \to \mathbb{C}$ and $Y \subseteq \mathbb{R} \to \mathbb{C}$ are set of signals. A system $H \in X \to Y$ with domain X is called **memoryless** if, for all input signals $x \in X$, the output signal Hx at time t depends only on x at time t. For example $\frac{1}{x(t)}$ with domain the set of signals that do not take the value zero and the identity system T_0 with domain the set of all signal $\mathbb{R} \to \mathbb{C}$ are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau \qquad x \in L_{\text{loc}}$$

are not. A shifter T_τ with $\tau \neq 0$ is not memoryless.

A system $H \in X \to Y$ is **causal** if, for all input signals $x \in X$, the output signal Hx at time t depends on x at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. The shifter T_τ is causal when $\tau \geq 0$, but is not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau \qquad x \in L_{\text{loc}}$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau)d\tau \qquad x \in L_{\text{loc}}$$

are not causal.

A system $H \in X \to Y$ is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal Hx is bounded whenever the input signal x is bounded. That is, H is stable if for every positive real number M there exists a positive real number K such that for all input signals $x \in X$ bounded below M, that is,

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it holds that the output signal Hx is bounded below K, that is,

$$|Hx(t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

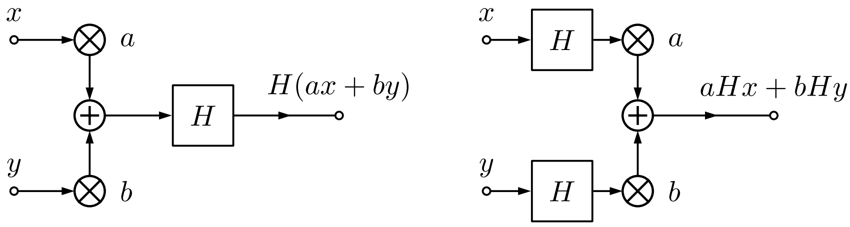

Figure 1.10: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax + by) = aHx + bHy$.

For example, the system $x(t) + 3x(t-1)$ with domain $\mathbb{R} \to \mathbb{C}$ is stable with $K = 4M$ since if $|x(t)| < M$, then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for $a \in \mathbb{R}$ having domain L_{loc} and the differentiator D with domain C^1 are not stable (Exercises 1.16 and 1.17).

Let $H \in X \to Y$ be a system with both the domain X and Y being linear spaces of signals. The system H is **linear** if

$$H(ax + by) = aHx + bHy$$

for all signals $x, y \in X$ and all complex numbers a and b. That is, a linear system has the property: if the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.10 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$D(ax + by)(t) = \frac{d}{dt}\big(ax(t) + by(t)\big) \quad = a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) = aDx(t) + bDy(t)$$

whenever both x and y are differentiable, that is, when $x, y \in C^1$. However, the system $Hx(t) = \frac{1}{x(t)}$ is not linear because

$$H(ax + by)(t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aHx(t) + bHy(t)$$

in general.

Let $H \in X \to Y$ be a system with both the domain X and Y being shift-invariant spaces. The system H is **shift-invariant** (or **time-invariant**) if

$$HT_\tau x(t) = Hx(t - \tau)$$

for all signals $x \in X$ and all shifts $\tau \in \mathbb{R}$. That is, a system is shift-invariant if shifting the input signal results in the same shift of the output signal. Equivalently, H is shift-invariant if it commutes with the shifter T_τ, that is, if

$$HT_\tau x = T_\tau Hx$$

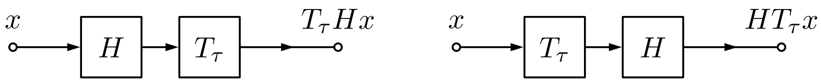

Figure 1.11: If H is a shift-invariant system the outputs of these two diagrams are the same signal, i.e. $HT_\tau x = T_\tau Hx$.

for all $\tau \in \mathbb{R}$ and all signals $x \in X$. Figure 1.11 represents the property of shift-invariance with a block diagram. For example, the differentiator is shift-invariant because

$$DT_\tau x(t) = \frac{d}{dt}x(t-\tau) = T_\tau Dx(t)$$

by the chain rule for differentiation. The integrator I_∞ and shift-invariant (Exercise 1.20) but the integrator I_a for finite $a < \infty$ is not (Exercise 1.19).

We will be primarily interested in systems that are both linear and shift-invariant. Such systems are said to be **linear shift-invariant** or **linear time-invariant** systems. The phrase "Let $H \in X \to Y$ be a linear shift-invariant system" will occur regularly and it will always imply that both sets X and Y are linear and shift-invariant spaces of signals.

Exercises

1.1. How many distinct functions from the set $X = \{\text{Mario}, \text{Link}\}$ to the set $Y = \{\text{Freeman}, \text{Ryu}, \text{Sephiroth}\}$ exist? Write down each function, that is, write down all functions from the set $X \to Y$.

1.2. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal.

1.3. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

1.6. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 5 and a change of variables).

1.7. Show that the signal t^2 is differentiable, but the step function u and rectangular pulse Π are not.

1.8. Plot the signal $\sin(t) + \sin(\pi t)$. Show that this signal is not periodic.

1.9. Show that the set of locally integrable signals L_{loc}, the set of absolutely integrable signals L^1, and the set of square integrable signals L^2 are linear shift-invariant spaces.

1.10. Show that the set of periodic signals is a shift-invariant space, but not a linear space.

1.11. Show that the set of bounded signals is a linear shift-invariant space.

1.12. Let $K > 0$ be a fixed real number. Show that the set of signals bounded below K is a shift invariant space, but not a linear space.

1.13. Show that the set of even signals and the set of odd signals are not shift invariant spaces.

1.14. Show that the integrator I_c with finite $c \in \mathbb{R}$ is not stable.

1.15. Show that if the signal x is locally integrable and $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ then $I_\infty x(t) = \int_{-\infty}^{t} x(t) dt < \infty$ for all $t \in \mathbb{R}$.

1.16. Show that the integrator I_∞ is not stable.

1.17. Show that the differentiator system D is not stable.

1.18. Show that the shifter T_τ is linear and shift-invariant and that the time-scaler is linear, but not time invariant.

1.19. Show that the integrator I_c with finite $c \in \mathbb{R}$ is linear, but not shift-invariant.

1.20. Show that the integrator I_∞ is linear and shift-invariant.

1.21. State whether the system $Hx = x + 1$ is linear, shift-invariant, stable.

1.22. State whether the system $Hx = 0$ is linear, shift-invariant, stable.

1.23. State whether the system $Hx = 1$ is linear, shift-invariant, stable.

1.24. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is neither absolutely integrable nor square integrable.

Chapter 2

Systems modelled by differential equations

Systems of particular interest are those where the input signal x and output signal y are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^{m} a_\ell \frac{d^\ell}{dt^\ell} x(t) = \sum_{\ell=0}^{k} b_\ell \frac{d^\ell}{dt^\ell} y(t),$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are real or complex numbers. In what follows we use the differentiator system D rather than the notation $\frac{d}{dt}$ to represent differentiation. To represent the ℓth derivative we write D^ℓ instead of $\frac{d^\ell}{dt^\ell}$. Using this notation the differential equation above is

$$\sum_{\ell=0}^{m} a_\ell D^\ell x = \sum_{\ell=0}^{k} b_\ell D^\ell y. \tag{2.0.1}$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 2.1. Let the signal v_R represent the voltage over the resistor and i the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri \qquad \text{and} \qquad i = CDy.$$

Combining these equations,

$$x = y + RCDy \tag{2.0.2}$$

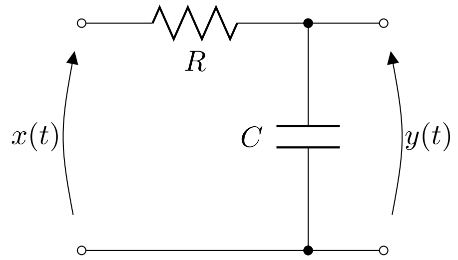

Figure 2.1: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

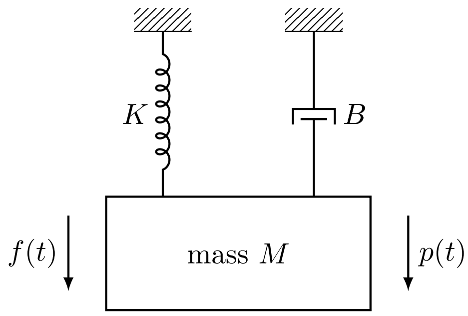

Figure 2.2: A mechanical mass-spring-damper system

that is in the form of (2.0.1).

As another example, consider the mass-spring-damper in Figure 2.2. A force represented by the signal f is externally applied to the mass, and the position of the mass is represented by the signal p. The spring exerts force $-Kp$ that is proportional to the position of the mass, and the damper exerts force $-BDp$ that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BDp$$

and by Newton's law the acceleration of the mass D^2p satisfies

$$MD^2p = f_m = f - Kp - BDp.$$

We obtain the differential equation

$$f = Kp + BDp + MD^2p \tag{2.0.3}$$

that is in the form of (2.0.1) if we put $x = f$ and $y = p$. Given p we can readily solve for the corresponding force f. As a concrete example, let the spring constant, damping constant and mass be $K = B = M = 1$. If the position satisfies $p(t) = e^{-t^2}$, then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

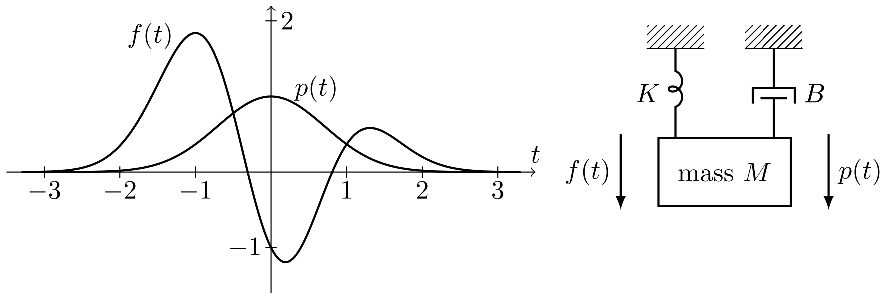

Figure 2.3: A solution to the mass-spring-damper system with $K = B = M = 1$. The position is $p(t) = e^{-t^2}$ with corresponding force $f(t) = e^{-t^2}(4t^2 - 2t - 1)$.

Figure 2.3 depicts these signals.

What happens if a particular force signal f is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \tfrac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal p? We are not yet ready to answer this question, but will be later (Exercise 4.14).

In both the mechanical mass-spring-damper system in Figure 2.2 and the electrical RC circuit in Figure 2.1 we obtain a differential equation relating the input signal x with the output signal y. The equations do not specify the output signal y explicitly in terms of the input signal x, that is, they do not explicitly define a system H such $y = Hx$. As they are, the differential equations do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? Much more information about these systems will be obtained when the **Laplace transform** is introduced in Chapter 4. The remainder of this chapter details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. The systems constructed will be used as examples throughout the text.

2.1 Passive electrical circuits

Passive electrical circuits require no sources of power other than the input signal itself. For example, the voltage divider in Figure 1.6 and the RC circuit in Figure 2.1 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 2.4. In this circuit we let the output signal y be the voltage over the resistor. Let v_C represent the voltage over the capacitor and v_L the voltage over the inductor and let i be the current. We have

$$y = Ri, \qquad i = CDv_C, \qquad v_L = LDi,$$

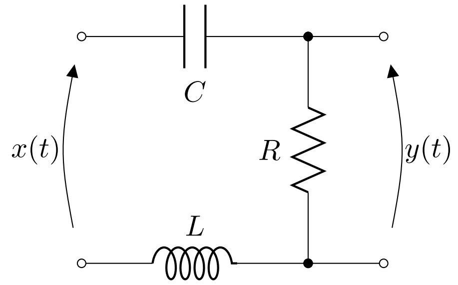

Figure 2.4: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

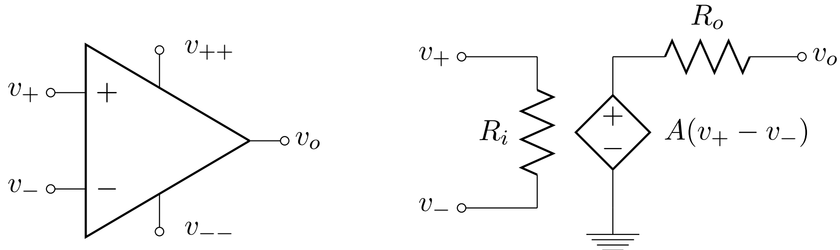

Figure 2.5: Left: triangular component diagram of an **operational amplifier**. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance R_i, output resistance R_o, and open loop gain A. The diamond shaped component is a dependent voltage source. This model is usually only useful when the operational amplifier is in a negative feedback circuit.

leading to the following relationships between y, v_C and v_L,

$$y = RCDv_C, \qquad Rv_L = LDy.$$

Kirchhoff's voltage law gives $x = y + v_C + v_L$ and by differentiating both sides

$$Dx = Dy + Dv_C + Dv_L.$$

Substituting the equations relating y, v_C and v_L leads to

$$RCDx = y + RCDy + LCD^2y. \tag{2.1.1}$$

We can similarly find equations relating the input voltage with v_C and v_L.

2.2 Active electrical circuits

Unlike passive electrical circuits, an **active electrical circuit** requires a source of power external to the input signal. Active circuits can be modelled and constructed using **operational amplifiers** as depicted in Figure 2.5. The left hand side of Figure 2.5 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 2.5 shows a circuit that can be used to model the behaviour of the amplifier. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. The diamond shaped component is a dependent voltage source with voltage $A(v_+ - v_-)$ that depends on the difference between the voltage at the **non-inverting input** v_+ and the voltage at the **inverting input** v_-. The dimensionless constant A is called the **open loop gain**. Most operational amplifiers have large open loop gain A, large **input resistance** R_i and small **output resistance** R_o. As we will see, it can be convenient to consider the behaviour as $A \to \infty$, $R_i \to \infty$ and $R_0 \to 0$, resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is depicted in Figure 2.6. This circuit is an example of an operational amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input v_-. The horizontal wire at the bottom of the plot is considered to be ground (zero volts) and is connected to the negative terminal of the dependent voltage source of the operational amplifier depicted in Figure 2.5. An equivalent circuit for the multiplier using the model in Figure 2.5 is shown in Figure 2.7. Solving this circuit (Exercise 2.1) yields the following relationship between the input voltage signal x and the output voltage signal y,

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x. \tag{2.2.1}$$

For an ideal operational amplifier we let $A \to \infty$, $R_i \to \infty$ and $R_o \to 0$. In this case terms involving the product AR_i dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1} x. \tag{2.2.2}$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant $-\frac{R_2}{R_1}$.

The equation relating x and y is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,

2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.2.2) using these rules. Because the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor R_2 is $y = R_2 i$. Because no current flows through the inverting input the current through R_1 is also i and $x = -R_1 i$. Combining these results, the input voltage x and the output voltage y are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisting of an operational amplifier, two resistors and two capacitors depicted in Figure 2.8. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor C_2 and resistor R_2 is equal to y and, by Kirchoff's current law,

$$i = \frac{y}{R_2} + C_2 D y.$$

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1 D x.$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1 D x = \frac{y}{R_2} + C_2 D y. \tag{2.2.3}$$

Observe the similarity between this equation and that for the passive RC circuit (2.0.2) when $R_1 = R_2$ and $C_1 = 0$ (an open circuit). In this case

$$x = -y - R_1 C_2 D y. \tag{2.2.4}$$

We call this the **active RC circuit**. This circuit is tested in Test 3.

Consider the circuit in Figure 2.9. Assuming an ideal operational amplifier, the input voltage x satisfies

$$-i = \frac{x}{R_1} + C_1 D x.$$

The voltage over the capacitor C_2 is $y - R_2 i$ and so the current satisfies

$$i = C_2 D(y - R_2 i).$$

Combining these equations gives

$$-\frac{x}{R_1} - C_1 D x = C_2 D y + \frac{R_2 C_2}{R_1} D x + R_2 C_2 C_1 D^2 x,$$

and after rearranging,

$$D y = -\frac{1}{R_1 C_2} x - \left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right) D x - R_2 C_1 D^2 x.$$

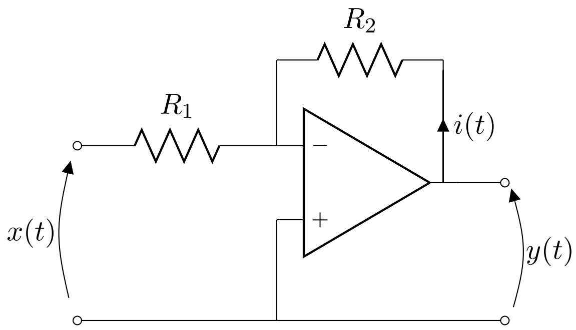

Figure 2.6: Inverting amplifier

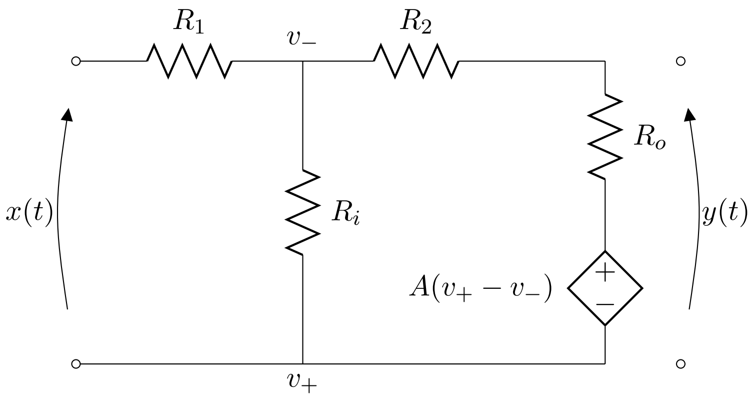

Figure 2.7: An equivalent circuit for the inverting amplifier from Figure 2.6 using the model for an operational amplifier in Figure 2.5.

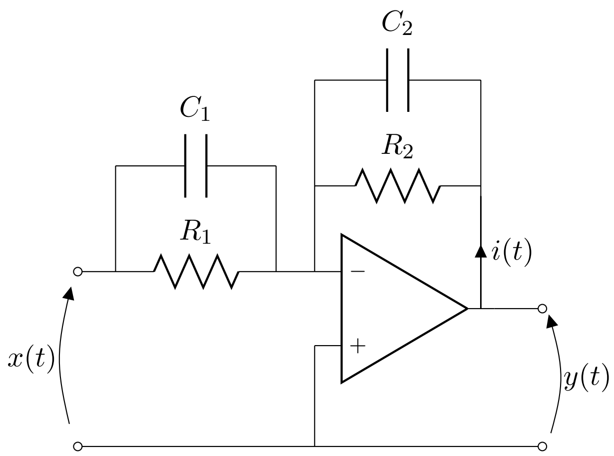

Figure 2.8: Operational amplifier configured with two capacitors and two resistors.

Test 2 (**Inverting amplifier**) In this test we construct the inverting amplifier circuit from Figure 2.6 with $R_2 \approx 22\text{k}\Omega$ and $R_1 \approx 12\text{k}\Omega$ that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 100$ and $f_2 = 233$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y. Approximate reconstructions of the input signal $\tilde{x}$ and output signal $\tilde{y}$ are given according to (1.3.4) and (1.3.2). According to (2.1.1) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1}\tilde{x} = -\frac{11}{6}\tilde{x}.$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\frac{11}{6}\tilde{x}$ are plotted in Figure 2.9.

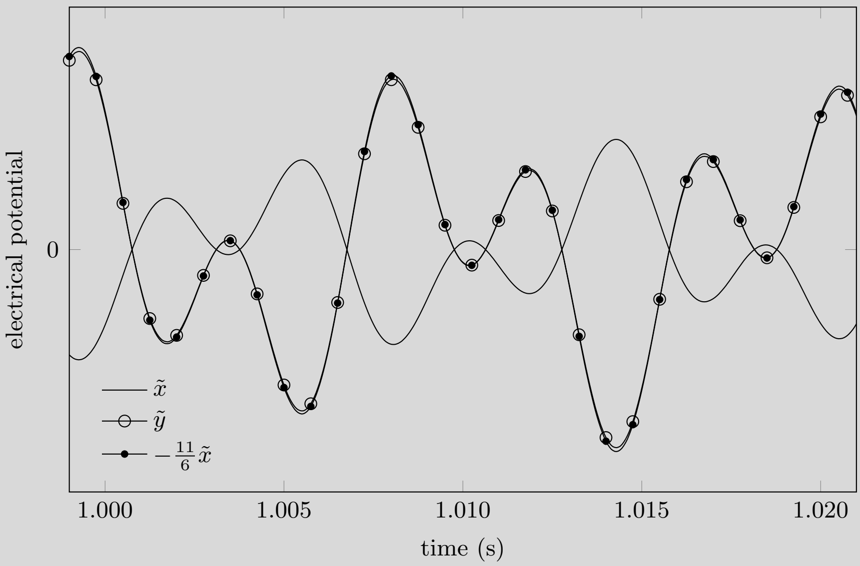

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $-\frac{11}{6}\tilde{x}$ (solid line with dot).

Test 3 (**Active RC circuit**) In this test we construct the circuit from Figure 2.8 with $R_1 \approx R_2 \approx 27\text{k}\Omega$ and $C_2 \approx 10\text{nF}$ accurate to within 5% of these values and $C_1 = 0$ (an open circuit). The operational amplifier used is a Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 500$ and $f_2 = 1333$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y and approximate reconstructions $\tilde{x}$ and $\tilde{y}$ are given according to (1.3.4) and (1.3.2). According to (2.2.4) we expect the approximate relationship

$$\tilde{x} \approx -\frac{R_1}{R_2}\tilde{y} - R_1 C D(\tilde{y}) = -\tilde{y} - \frac{27}{10^5} D(\tilde{y}).$$

The derivative of the sinc function is

$$D\,\text{sinc}(t) = \frac{d}{dt}\text{sinc}(t) = \frac{1}{\pi t^2}\big(\pi t\cos(\pi t) - \sin(\pi t)\big), \qquad (2.2.5)$$

and so,

$$D\tilde{y}(t) = \frac{d}{dt}\left(\sum_{\ell=1}^{L} y_\ell\,\text{sinc}(Ft - \ell)\right) = F\sum_{\ell=1}^{L} y_\ell D\,\text{sinc}(Ft - \ell). \qquad (2.2.6)$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\tilde{y} - \frac{27}{10^5}D\tilde{y}$ are plotted in Figure 2.9.

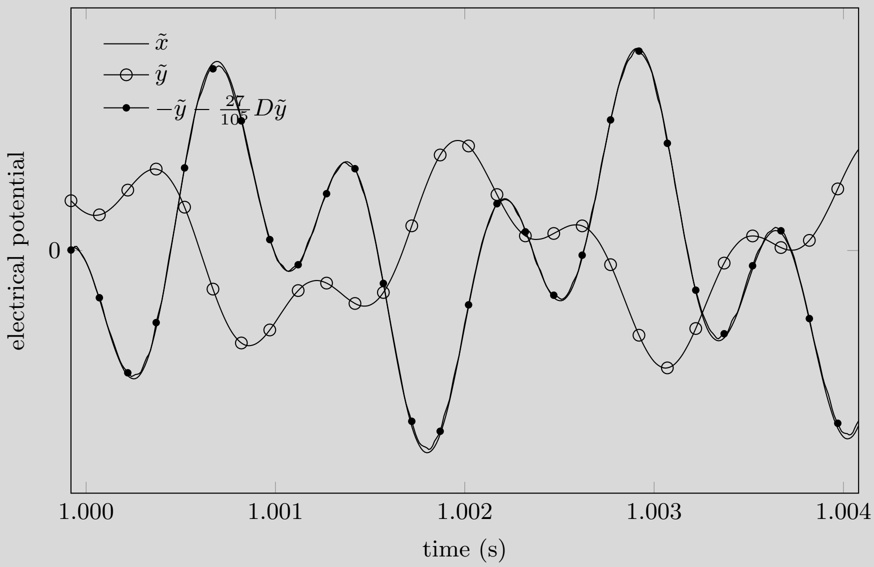

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line with circle), output signal $\tilde{y}$ (solid line), and hypothesised input signal $-\tilde{y} - \frac{27}{10^5}D\tilde{y}$ (solid line with dot).

Put

$$K_i = \frac{1}{R_1 C_2}, \qquad K_p = \frac{R_2}{R_1} + \frac{C_1}{C_2}, \qquad K_d = R_2 C_1$$

and now

$$Dy = -K_i x - K_p Dx - K_d D^2 x. \tag{2.2.7}$$

This equation models what is called a **proportional-integral-derivative controller** or **PID controller**. The coefficients K_i, K_p and K_d are called the **integral gain**, **proportional gain**, and **derivative gain**.

The final active circuit we consider is called a **Sallen-Key** [Sallen and Key, 1955] and is depicted in Figure 2.10. Observe that the output of the amplifier is connected directly to the inverting input and is also connected to the noninverting input by a capacitor and resistor. This circuit has both negative *and* positive feedback. It is not immediately apparent that we can use the simplifying assumptions for an ideal operational amplifier with negative feedback. However, we will do so and will find that it works in this case.

Let v_{R1}, v_{R2}, v_{C1}, and v_{C2} be the voltages over the components R_1, R_2, C_1, and C_2. Kirchoff's voltage law leads to the equations

$$x = v_{R1} + v_{R2} + v_{C2}, \qquad y = v_{C1} + v_{R2} + v_{C2}.$$

The voltage at the inverting and noninverting terminals is y and so the voltage over the capacitor C_2 is y, that is, $y = v_{C2}$. Using this, the equations above simplify to

$$x = v_{R1} + v_{R2} + y, \qquad v_{C1} = -v_{R2}.$$

The current i_2 through capacitor C_2 satisfies $i_2 = C_2 D v_{C2} = C_2 D y$. Because no current flows into the inverting terminal of the amplifier the current through R_2 is also i_2 and so $v_{R2} = R_2 i_2 = R_2 C_2 D y$. Substituting this into the equations above gives

$$x = v_{R1} + R_2 C_2 D y + y, \qquad v_{C1} = -R_2 C_2 D y. \tag{2.2.8}$$

Kirchoff's current law asserts that $i + i_1 = i_2$. The current i through capacitor C_1 satisfies $i = C_1 D v_{C1} = -R_2 C_1 C_2 D^2 y$ and the current through resistor R_1 satisfies

$$v_{R1} = R_1 i_1 = R_1(i_2 - i) = R_1 C_2 D y + R_1 R_2 C_1 C_2 D^2 y.$$

Substituting this into the equation on the left of (2.2.8) gives

$$x = y + C_2(R_1 + R_2) D y + R_1 R_2 C_1 C_2 D^2 y. \tag{2.2.9}$$

The Sallen-Key will be useful when we consider the design of analogue electrical filters in Section 5.2.

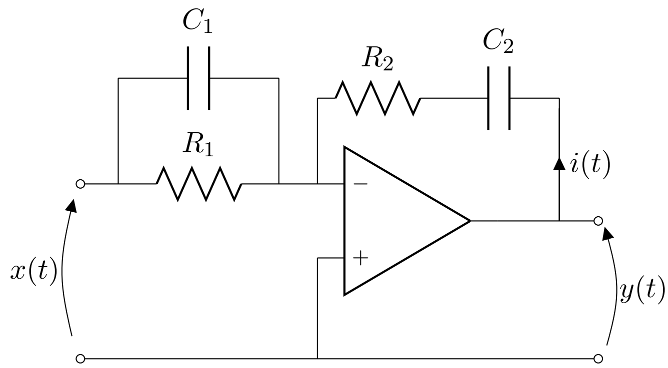

Figure 2.9: Operational amplifier implementing a **proportional-integral-derivative controller**.

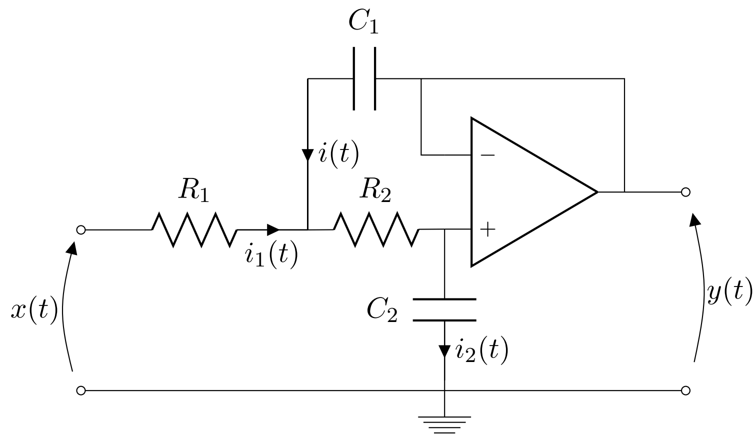

Figure 2.10: Operational amplifier implementing a **Sallen-Key**.

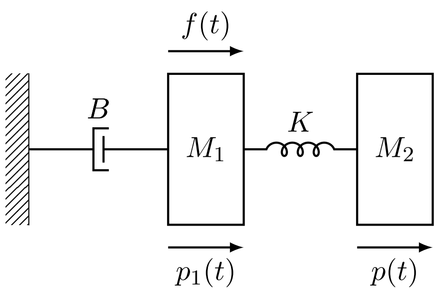

Figure 2.11: Two masses, a spring and a damper

2.3 Masses, springs, and dampers

A mechanical mass-spring-damper system was described in Section 2 and Figure 2.2. We now consider another mechanical system depicted in Figure 2.11 involving two masses, a spring and a damper. A mass M_1 is connected to a wall by a damper with constant B, and to another mass M_2 by a spring with constant K. A force represented by the signal f is applied to the first mass. We will derive a differential equation relating f with the position p of the second mass. Assume that the spring applies no force (is in equilibrium) when the masses are distance d apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where f_{s1} and f_{s2} are signals representing the force due to the spring on mass M_1 and M_2 respectively. It is convenient to define the signal $g = p_1 + d$ so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to M_2 is by the spring and so, by Newton's law, the acceleration of M_2 satisfies

$$M_2 D^2 p = f_{s2} = Kg - Kp. \tag{2.3.1}$$

The force applied by the damper on mass M_1 is given by the signal

$$f_d = -BDp_1 = -BDg$$

where the replacement of p_1 by g is justified because differentiation will remove the constant d. The cumulative force on M_1 is given by the signal

$$f_1 = f + f_d + f_{s1} = f + f_d - f_{s2} = f - M_2 D^2 p - BDg$$

and by Newton's law the acceleration of M_1 satisfies

$$M_1 D^2 p_1 = M_1 D^2 g = f_1 = f - M_2 D^2 p - BDg.$$

Combining this equation with (2.3.1) we obtain a fourth order differential equation relating the position p and force f,

$$f = BDp + (M_1 + M_2)D^2 p + \frac{BM_2}{K} D^3 p + \frac{M_1 M_2}{K} D^4 p. \tag{2.3.2}$$

Given the position of the second mass p we can readily solve for the corresponding force f and position of the first mass p. For example, if the constants $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and $d = \frac{5}{2}$, and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.3.2) and (2.3.1),

$$f(t) = e^{-t^2}(1 + 4t - 8t^2 - 4t^3 + 4t^4), \qquad \text{and} \qquad p_1(t) = 2e^{-t^2}t^2 - \tfrac{5}{2}.$$

This solution is plotted in Figure 2.12.

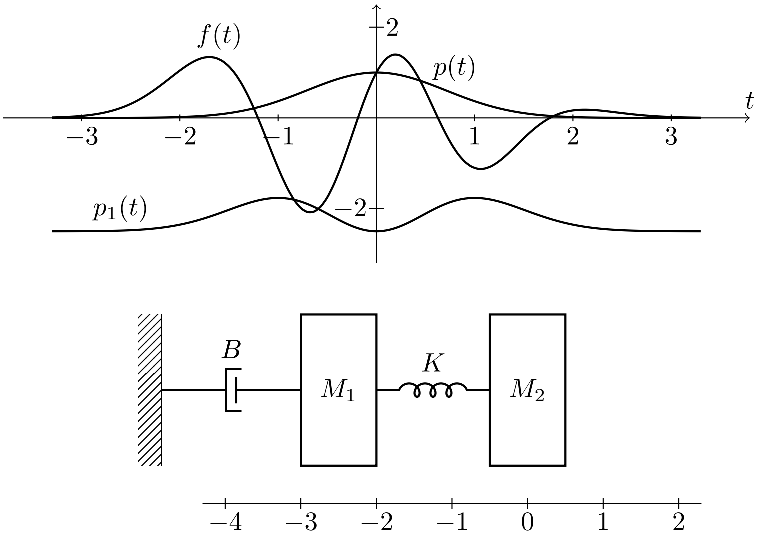

Figure 2.12: Solution of the system describing two masses with a spring and damper where $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and the position of the second mass is $p(t) = e^{-t^2}$.

2.4 Direct current motors

Direct current (DC) motors convert electrical energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage v to the angular position of the motor θ. Figure 2.13 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \qquad v_L = LDi,$$

and the motion of the motor induces a voltage called the **back electromotive force** (EMF),

$$v_b = K_b D\theta$$

that we model as being proportional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LDi + K_b D\theta.$$

The torque τ applied by the motor is modelled as being proportional to the current i,

$$\tau = K_\tau i.$$

A load with inertia J is attached to the motor. Two forces are assumed to act on the load, the torque τ applied by the current, and a torque $\tau_d = -BD\theta$

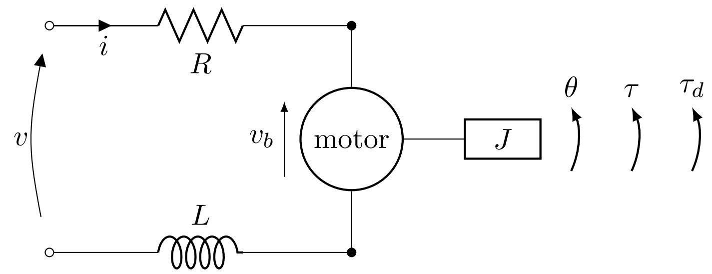

Figure 2.13: Diagram for a rotary direct current (DC) motor

modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD^2\theta = \tau + \tau_d = K_\tau i - BD\theta.$$

Combining these equations we obtain the 3rd order differential equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ + LB}{K_\tau} D^2\theta + \frac{LJ}{K_\tau} D^3\theta$$

relating voltage and motor position. In many DC motors the inductance L is small and can be ignored, leaving the simpler second order equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta. \tag{2.4.1}$$

Given the position signal θ we can find the corresponding voltage signal v. For example, put the constants $K_b = K_\tau = B = R = J = 1$ and assume that

$$\theta(t) = 2\pi(1 + \text{erf}(t))$$

where $\text{erf}(t) = \frac{2}{\pi}\int_0^t e^{-\tau^2} d\tau$ is the **error function**. The corresponding angular velocity $D\theta$ and voltage v satisfy

$$D\theta(t) = 4\sqrt{\pi}e^{-t^2}, \qquad v(t) = 8\sqrt{\pi}e^{-t^2}(1 - t).$$

These signals are depicted in Figure 2.14. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Maxima, Sage, Mathematica, or Maple).

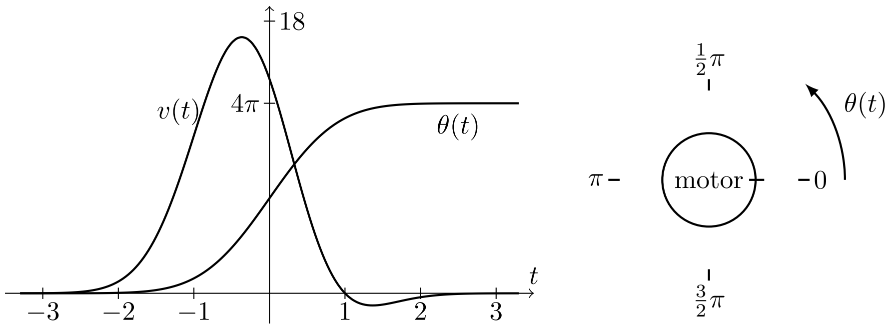

Figure 2.14: Voltage and corresponding angle for a DC motor with constants $K_b = K_\tau = B = R = J = 1$.

2.2. Figure 2.15 depicts a mechanical system involving two masses, two springs, and a damper connected between two walls. Suppose that the spring K_2 is at rest when the mass M_2 is at position $p(t) = 0$. A force, represented by the signal f, is applied to mass M_1. Derive a differential equation relating the force f and the position p of mass M_2. Determine the force f in the case that the position $p(t) = e^{-t^2}$ and $M_1 = M_2 = \frac{1}{2}$ and $K_1 = K_2 = B = 1$.

2.3. Consider the electromechanical system in Figure 2.16. A direct current motor is connected to a potentiometer in such a way that the voltage at the output of the potentiometer is equal to the angle of the motor θ. This voltage is fed back via a unity gain amplifier to the input terminal of the motor. An input voltage v is applied to the other terminal on the motor. Find the differential equation relating v and θ. What is the input voltage v if the motor angle satisfies $\theta(t) = \frac{\pi}{2}(1 + \text{erf}(t))$? Plot θ and v in this case when the motor coefficients satisfy $L = 0$, $R = \frac{3}{4}$, and $K_b = K_\tau = B = J = 1$.

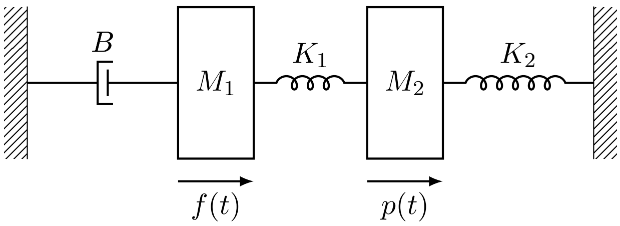

Figure 2.15: Two masses, a spring, and a damper connected between two walls for Exercise 2.2.

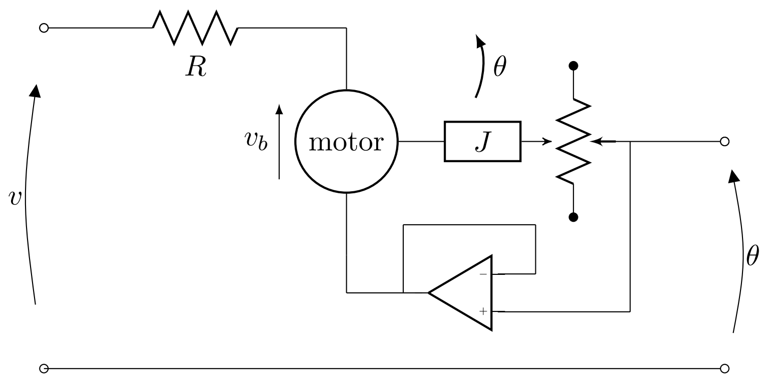

Figure 2.16: Diagram for a rotary direct current (DC) with potentiometer feedback for Exercise 2.3.

Chapter 3

Linear shift-invariant systems

In the previous section we derived differential equations that model mechanical, electrical, and electro-mechanical systems. The equations themselves often do not provide sufficient information. For example, we were able to find a signal p representing the position of the mass-spring-damper in Figure 2.2 given a particular force signal f is applied to the mass. However, it is not immediately obvious how to find the force signal f given a particular position signal p. We will be able to solve this problem and, more generally, to describe properties of systems modelled by linear differential equations with constant coefficient, if we make the added assumptions that the systems are **linear** and **shift-invariant**. We study linear shift-invariant systems in this chapter.

3.1 Convolution, regular systems and the delta "function"

A large number of linear shift-invariant systems can be represented by a signal called the **impulse response**. The impulse response of a system H is a locally integrable signal h such that

$$Hx(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau,$$

that is, the response of H to input signal x can be represented as an integral equation involving x and the impulse response h. The integral is called a **convolution** and appears so often that a special notation is used for it. We write $h * x$ to indicate the signal that results from convolution of signals h and x, that is, $h * x$ is the signal

$$h * x = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

where the right hand side is to be interpreted as a signal, a function of $t \in \mathbb{R}$. We write $(h * x)(t)$ to indicate the value of $h * x$ at $t \in \mathbb{R}$. Those systems that have an impulse response we call **regular systems**[1].

Some care must be taken when selecting a domain for a regular system. To see what can go wrong it is worth first considering an example. Suppose that H has discrete impulse response given by the step function u (1.1.1). The signal $x(t) = 1$ that takes the value 1 for all $t \in \mathbb{R}$ will not be in the domain of H since, in this case,

$$Hx(t) = \int_{-\infty}^{\infty} u(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} d\tau$$

is not finite for any $t \in \mathbb{R}$. Given a signal h, denote by $\operatorname{dom} h$ the set of signals x such that the integral

$$\int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

If H has impulse reponse h then, for all signal $x \in \operatorname{dom} h$ we have

$$|Hx(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau < \infty$$

for all $t \in \mathbb{R}$ and so $Hx(t)$ is finite for all $t \in \mathbb{R}$. We take $\operatorname{dom} h$ as the domain of a regular system H with impulse response h unless otherwise stated. It can be shown that $\operatorname{dom} h$ is a linear shift-invariance space (Exercise 3.3).

Regular systems are linear because, for all $x, y \in \operatorname{dom} h$ and all $a, b \in \mathbb{C}$,

$$\begin{aligned} H(ax+by) &= h * (ax+by) \\ &= \int_{-\infty}^{\infty} h(\tau)\big(ax(t-\tau) + by(t-\tau)\big)d\tau \\ &= a\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + b\int_{-\infty}^{\infty} h(\tau)y(t-\tau)d\tau \\ &= a(h*x) + b(h*y) \\ &= aHx + bHy. \end{aligned} \tag{3.1.1}$$

The above equation shows that convolution commutes with scalar multiplication and distributes with addition, that is,

$$h * (ax+by) = a(h*x) + b(h*y).$$

[1] The name **regular system** is motivated by the term **regular distribution** [Zemanian, 1965]

Regular systems are also shift-invariant because for all $x \in \operatorname{dom} h$

$$\begin{aligned} T_\kappa H x &= T_\kappa (h * x) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t - \kappa - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) T_\kappa x(t - \tau) d\tau \\ &= h * (T_\kappa x) \\ &= H T_\kappa x. \end{aligned}$$

The impulse response of a regular system H can be found in the following way. First define the signal

$$p_\gamma(t) = \begin{cases} \gamma, & 0 < t \leq \frac{1}{\gamma} \\ 0, & \text{otherwise}, \end{cases}$$

that is, a rectangular shaped pulse of height γ and width $\frac{1}{\gamma}$. The signal p_γ is plotted in Figure 3.1 for $\gamma = \frac{1}{2}, 1, 2, 5$. As γ increases the pulse gets thinner and higher so as to keep the area under p_γ equal to one. Consider the response of the regular system H to the signal p_γ,

$$H p_\gamma = h * p_\gamma = \int_{-\infty}^{\infty} h(\tau) p_\gamma(t - \tau) d\tau = \gamma \int_{t-1/\gamma}^{t} h(\tau) d\tau.$$

Taking $\gamma \to \infty$ we find that

$$\lim_{\gamma \to \infty} H p_\gamma = \lim_{\gamma \to \infty} \gamma \int_{t-1/\gamma}^{t} h(\tau) d\tau = h \quad \text{a.e.}$$

As an example, consider the integrator system I_∞ described in Section 1.4. The response of I_∞ to p_γ is

$$I_\infty p_\gamma(t) = \int_{-\infty}^{t} p_\gamma(\tau) d\tau = \begin{cases} 0, & t \leq 0 \\ \gamma t, & 0 < t \leq \frac{1}{\gamma} \\ 1, & t > \frac{1}{\gamma}. \end{cases}$$

The response is plotted in Figure 3.1. Taking the limit as $\gamma \to \infty$ we find that the impulse response of the integrator is the step function

$$u(t) = \lim_{\gamma \to \infty} H p_\gamma(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases} \quad \text{a.e.} \tag{3.1.2}$$

Some important systems do not have an impulse response and are not regular. For example, the identity system T_0 is not regular. Similarly, the

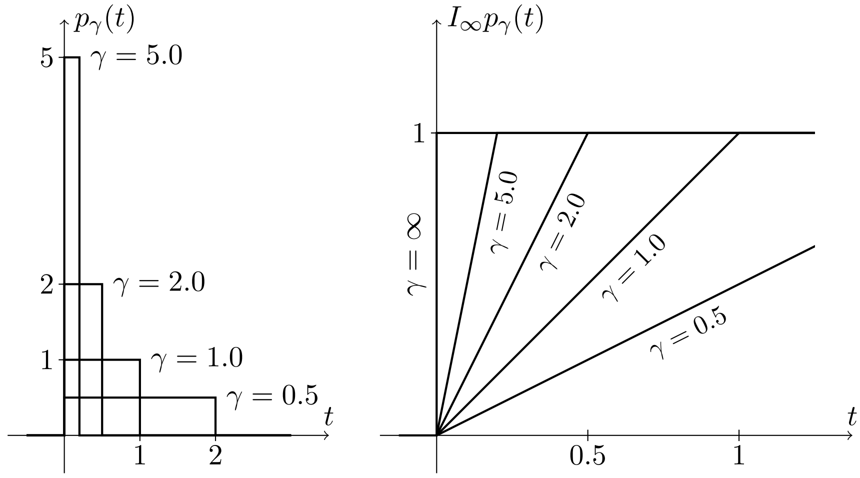

Figure 3.1: The rectangular shaped pulse p_γ for $\gamma = 0.5, 1, 2, 5$ and the response of the integrator I_∞ to p_γ for $\gamma = 0.5, 1, 2, 5, \infty$.

shifter T_τ and differentiators D^k are not regular. However, it is common to pretend that T_0 *does* have an impulse response and this is typically denoted by the symbol δ called the **delta function**. The idea is to assign δ the property

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

so that convolution of x and δ satisfies

$$\delta * x = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = x(t) = T_0 x.$$

We now treat δ as if it were a signal. So $\delta(t-\tau)$ will represent the impulse response of the shifter T_τ because

$$\begin{aligned} T_\tau x &= \delta(t-\tau) * x \\ &= \int_{-\infty}^{\infty} \delta(\kappa-\tau)x(t-\kappa)d\kappa \\ &= \int_{-\infty}^{\infty} \delta(k)x(t-\tau-k)dk \qquad \text{(change variable } k = \kappa - \tau) \\ &= x(t-\tau). \end{aligned}$$

For $a \in \mathbb{R}$ it is common to plot $a\delta(t-\tau)$ using an arrow of height a at $t = \tau$ as indicated in Figure 3.2. It is important to realise that δ is not actually a signal. It is not a function in $\mathbb{R} \to \mathbb{C}$. However, it can be convenient to treat δ as if it were a signal. The manipulations in the last set of equations, such as the change of variables, are not formally justified, but they do lead to the desired result $T_\tau x = x(t-\tau)$ in this case. In general, there is no guarantee

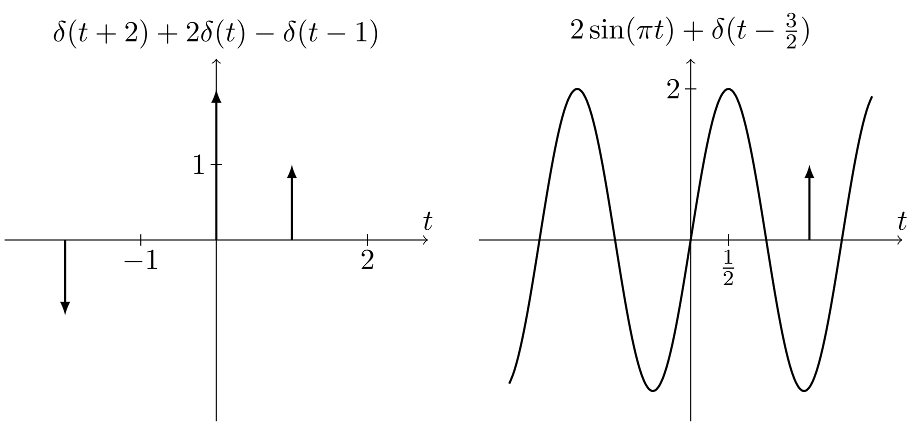

Figure 3.2: Plot of the "signal" $\delta(t+2) + 2\delta(t) - \delta(t-1)$ (left) and the "signal" $2\sin(\pi t) + \delta(t - \frac{3}{2})$ (right).

that mechanical mathematical manipulations involving δ will lead to sensible results.

The only other non regular systems that we have use of are differentiators D^k and it is common to define a similar notation for pretending that these systems have an impulse response. In this case, the symbol δ^k is assigned the property

$$\int_{-\infty}^{\infty} x(t)\delta^k(t)dt = D^k x(0),$$

so that convolution of x and δ^k is

$$\delta^k * x = \int_{-\infty}^{\infty} \delta^k(\tau)x(t-\tau)d\tau = D^k x(t).$$

As with the delta function the symbol δ^k must be treated with care. This notation can be useful, but purely formal manipulations with δ^k may not always lead to sensible results.

The impulse response h immediately yields some properties of the corresponding system H. For example, if $h(t) = 0$ for all $t < 0$, then H is causal because

$$Hx(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

only depends on values of the input signal x at $t-\tau$ with $\tau > 0$. The system H is stable if and only if h is absolutely integrable (Exercise 3.5).

Another related important signal is the **step response** defined as the response of the system to the step function u. For example, the step response of the shifter T_τ is the shifted step function $T_\tau u(t) = u(t-\tau)$. The step response of the integrator I_∞ is

$$I_\infty u(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} \int_0^t d\tau = t & t > 0 \\ 0 & t \leq 0. \end{cases}$$

This signal is often called the **ramp function**. A system has a step response only if u is inside its domain. For example, the regular system with impulse response $u(-t)$ does not have a step response because $u \notin \operatorname{dom} u(-t)$. Convolution of the step function u and its reflection $u(-t)$ is not possible. If a system H has both an impulse response h and a step response Hu, then these two signals are related. To see this, observe that the step response is

$$Hu = h * u = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = I_\infty h. \tag{3.1.3}$$

Thus, the step response can be obtained by applying the integrator I_∞ to the impulse response in the case that both of these signals exist.

3.2 Properties of convolution

The convolution $x * y$ of two signals x and y does not always exist. For example, if $x(t) = u(t)$ and $y(t) = 1$, then

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)d\tau = \int_{0}^{\infty} d\tau$$

is not finite for any t. We cannot convolve the step function u and the signal that takes the constant value 1. On the other hand, if $x(t) = y(t) = u(t)$, then

$$x * y = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t d\tau = \tau & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is finite for all t. If $x \in \operatorname{dom} y$ or equivalently $y \in \operatorname{dom} x$ then the convolution $x * y$ exists because, in this case,

$$|(x*y)(t)| = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \leq \int_{-\infty}^{\infty} |x(\tau)y(t-\tau)|\, d\tau < \infty$$

for all $t \in \mathbb{R}$.

We have already shown in (3.1.1) that convolution commutes with scalar multiplication and distributes with addition, that is, for signals x, y, w and complex numbers a, b,

$$a(x * w) + b(y * w) = (ax + by) * w.$$

The property holds provided that the convolutions $x * w$ and $y * w$ exist. This is the case if, for example, w is in both $\operatorname{dom} x$ and $\operatorname{dom} y$, that is,

$w \in \operatorname{dom} x \cap \operatorname{dom} y$. Convolution is commutative, that is, $x * y = y * x$ whenever these convolutions exist. To see this, write

$$\begin{aligned} x * y &= \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(t-\kappa) y(\kappa) d\kappa \qquad \text{(change variable } \kappa = t - \tau\text{)} \\ &= y * x. \end{aligned}$$

Convolution is associative under appropriate assumptions, that is, for signals x, y, z, we have $x * (y * z) = (x * y) * z$. To describe conditions under which associativity holds let us first define the set $\operatorname{dom}(x, y)$ containing all those signals z such that the double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau) y(\kappa) z(t - \kappa - \tau)| \, d\kappa \, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

We will often drop the brackets and write simply $\operatorname{dom} x\, y$. If $z \in \operatorname{dom} x\, y$ then

$$\begin{aligned} x * (y * z) &= \int_{-\infty}^{\infty} x(\tau)(y * z)(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(\kappa) z(t - \kappa - \tau) \, d\kappa \, d\tau. \end{aligned}$$

Because $z \in \operatorname{dom} x\, y$, Fubini's theorem [Rudin, 1986, Theorem 8.8] justifies swapping the order of integration leading to

$$x * (y * z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(\kappa) z(t - \kappa - \tau) \, d\tau \, d\kappa$$

and by the change of variable $\nu = \kappa + \tau$ we find that

$$\begin{aligned} x * (y * z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(\nu - \tau) z(t - \nu) \, d\tau \, d\nu \\ &= \int_{-\infty}^{\infty} (x * y)(\nu) z(t - \nu) \, d\nu \\ &= (x * y) * z. \end{aligned}$$

By combining the associative and commutative properties we find that the order in which the convolutions in $x*y*z$ are performed does not mater, that is

$$x * y * z = y * z * x = z * x * y = y * x * z = x * z * y = z * y * x$$

provided that all the convolutions involved exist. For example, this holds if $z \in \operatorname{dom} x\, y$ or equivalently $x \in \operatorname{dom} y\, z$ or $y \in \operatorname{dom} x\, z$. More generally, the order in which any sequence of convolutions is performed does not change the final result.

3.3 Linear combination and composition

Let F and G be linear shift-invariant systems with domain $X_F \subseteq \mathbb{R} \to \mathbb{C}$ and $X_G \subseteq \mathbb{R} \to \mathbb{C}$ respectively. For complex numbers c and d, let H be the system satisfying

$$Hx = cFx + dGx \qquad x \in X_F \cap X_G.$$

The system H is said to be a **linear combination** of F and G and its domain is taken to be $X_G \cap X_F$ unless otherwise stated. The system H is linear because for all signals $x, y \in X_G \cap X_F$ and $a, b \in \mathbb{C}$,

$$\begin{aligned} H(ax+by) &= cF(ax+by) + dG(ax+by) \\ &= acFx + bcFy + adGx + bdGy & \text{(linearity } F, G) \\ &= a(cFx + dGx) + b(cFy + dGy) \\ &= aHx + bHy. \end{aligned}$$

The system H is also shift-invariant because, for $x \in X_G \cap X_F$,

$$\begin{aligned} HT_\tau x &= cFT_\tau x + dGT_\tau x \\ &= cT_\tau Fx + dT_\tau Gx & \text{(shift-invariance } F, G) \\ &= T_\tau(cFx + dGx) & \text{(linearity } T_\tau) \\ &= T_\tau Hx. \end{aligned}$$

So, linear shift-invariant systems can be constructed as linear combinations of other linear shift-invariant systems. If F and G are regular systems with impulse responses f and g then, for $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\begin{aligned} Hx &= aFx + bGx \\ &= af * x + bg * x \\ &= (af + bg) * x & \text{(distributivity of convolution)} \\ &= h * x, \end{aligned}$$

and so, H is a regular system with impulse response $h = af + bg$. Its domain is taken to be $\operatorname{dom} f \cap \operatorname{dom} g$ unless otherwise stated.

Another way to construct linear shift-invariant systems is by **composition**. Let X, Y, Z be linear shift-invariant spaces of signals. Let $F \in X \to Y$ and $G \in Y \to Z$ be linear shift-invariant systems and let $H \in X \to Z$ be the system satisfying

$$Hx = GFx,$$

that is, H first applies F and then applies G. The system H is said to be the **composition** of F and G. Observe that the range of F is contained within the domain Y of G. This is necessary for the composition GF to make sense.

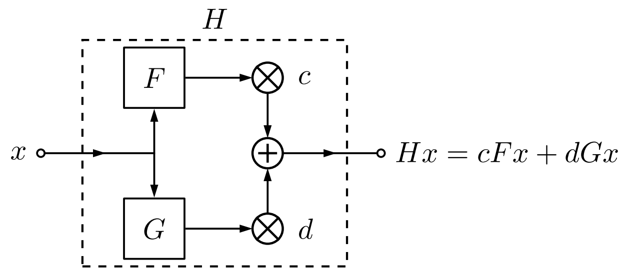

Figure 3.3: Block diagram depicting the linear combination of linear shift-invariant systems. The system $cFx + dGx$ can be expressed as a single linear shift-invariant system Hx.

The system H is linear because, for signals $x, y \in X$ and complex numbers a, b,

$$\begin{aligned} H(ax + by) &= GF(ax + by) \\ &= G(aFx + bFy) && \text{(linearity } F\text{)} \\ &= aGFx + bGFy && \text{(linearity } G\text{)} \\ &= aHx + bHy. \end{aligned}$$

The system is also shift-invariant because, for $x \in X$,

$$\begin{aligned} HT_\tau x &= GFT_\tau x \\ &= GT_\tau Fx && \text{(shift-invariance } F\text{)} \\ &= T_\tau GFx && \text{(shift-invariance } G\text{)} \\ &= T_\tau Hx. \end{aligned}$$

If F and G are regular systems the composition property can be expressed using their impulse responses f and g. For $x \in \operatorname{dom} f\ g$, the associative property of convolution asserts that $g * (f * x) = (g * f) * x$ (Section 3.2). It follows that, for $x \in \operatorname{dom} f\ g$,

$$Hx = GFx = g * (f * x) = (g * f) * x = h * x$$

and so H is a regular system with impulse response $h = g * f$. We can take its domain to be $\operatorname{dom} f\ g$ unless otherwise stated.

A wide variety of linear shift-invariant systems can be constructed by linear combination and composition of simpler systems.

3.4 Eigenfunctions and the transfer function

Let $s = \sigma + j\omega \in \mathbb{C}$ where $j = \sqrt{-1}$. A signal of the form

$$e^{st} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}\big(\cos(\omega t) + j \sin(\omega t)\big)$$

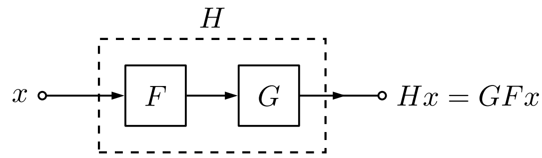

Figure 3.4: Block diagram depicting composition of linear shift-invariant systems. The system GF can be expressed as a single linear shift-invariant system H.

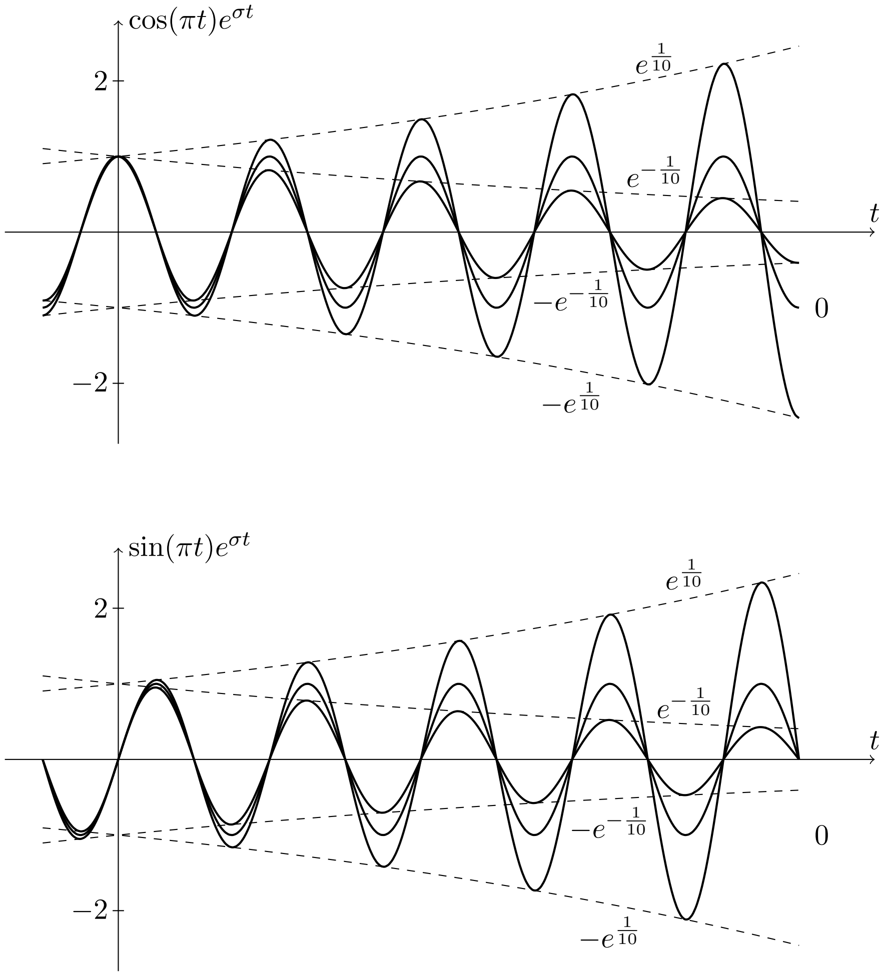

Figure 3.5: The function $\cos(\pi t)e^{\sigma t}$ (top) and $\sin(\pi t)e^{\sigma t}$ (bottom) for $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$.

is called a **complex exponential signal**. Complex exponential signals play an important role in the study of linear shift-invariant systems. The real and imaginary parts of $e^{(\sigma+j\pi)t}$ are plotted in Figure 3.5 for $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$. The signal is oscillatory when $\omega \neq 0$. The signal converges to zero as $t \to \infty$ when $\sigma < 0$ and diverges as $t \to \infty$ when $\sigma > 0$.

Let $H \in X \to Y$ be a linear shift-invariant system. Suppose that $y = He^{st}$ is the response of H to the complex exponential signal $e^{st} \in X$. Consider the response of H to the shifted signal $T_\tau e^{st} = e^{s(t-\tau)}$ for $\tau \in \mathbb{R}$. By shift-invariance

$$HT_\tau e^{st} = T_\tau He^{st} = y(t-\tau)$$

and by linearity

$$HT_\tau e^{st} = He^{s(t-\tau)} = e^{-s\tau}He^{st} = e^{-s\tau}y(t).$$

Combining these equations we obtain

$$y(t-\tau) = e^{-s\tau}y(t) \qquad \text{for all } t, \tau \in \mathbb{R}.$$

This equation is satisfied by signals of the form $y(t) = \lambda e^{st}$ where λ is a complex number. That is, the response of a linear shift-invariant system H to a complex exponential signal e^{st} is the same signal e^{st} multiplied by some constant complex number λ. Due to this property complex exponential signals are called **eigenfunctions** of linear shift-invariant systems. The constant λ does not depend on t, but it does usually depend on the complex number s and the system H. To highlight this dependence on H and s we write $\lambda H(s)$ or $\lambda(H)(s)$ or $\lambda(H,s)$ and do not distinguish between these notations. Considered as a function of s, the expression λH is called the **transfer function** of the system H. Observe that the transfer function λH maps a complex number to a complex number.

Denote by $\operatorname{cep} X$ the set of complex numbers s such that $e^{st} \in X$,

$$\operatorname{cep} X = \{s \in \mathbb{C} \,;\, e^{st} \in X\}.$$

We take $\operatorname{cep} X$ as the domain of the transfer function λH, that is $\lambda H \in \operatorname{cep} X \to \mathbb{C}$. The transfer function satisfies

$$He^{st} = \lambda H(s)e^{st} \qquad s \in \operatorname{cep} X. \tag{3.4.1}$$

This is an important equation. Stated in words: the response of a linear shift-invariant system $H \in X \to Y$ to a complex exponential signal $e^{st} \in X$ is the transfer function λH evaluated at s multiplied by the same complex exponential signal e^{st}.

We can use these eigenfunctions to better understand the properties of systems modelled by differential equations, such as those in Section 2. Consider the active electrical circuit from Figure 2.8. In the case that the resistors $R_1 = R_2$, and the capacitor $C_1 = 0$ (an open circuit) the differential equation relating the input voltage x and output voltage y is

$$x = -y - R_1 C_2 D y.$$

We called this the **active RC** circuit. To simplify notation put $R = R_1$ and $C = C_2$ so that $x = -y - RCDy$. Suppose that H is a linear shift-invariant system that maps the input voltage x to the output voltage y, that is, H is a system that describes the active RC circuit. If the input voltage is a complex exponential signal $x = e^{st}$, then the output voltage is the same complex exponential signal multiplied by the transfer function, that is, $y = Hx = \lambda H(s) e^{st}$. Substituting this into the differential equation for the active RC circuit we obtain

$$e^{st} = -\lambda e^{st} - RCD(\lambda e^{st}) = -\lambda e^{st}(1 - RCs)$$

where, to simplify notation, we have written simply λ for $\lambda H(s)$ above. Solving for λ leads to the transfer function of the system H describing the active RC circuit

$$\lambda H(s) = -\frac{1}{1 + RCs}. \tag{3.4.2}$$

Now, if e^{st} is input to the circuit, we expect the output to be

$$He^{st} = \lambda H(s) e^{st} = -\frac{e^{st}}{1 + RCs}.$$

This satisfies the differential equation $x = -y - RCDy$ for the active RC circuit.

In Chapter 2 we modelled electrical, mechanical, and electromechanical devices by differential equations of the form

$$\sum_{\ell=0}^{m} a_\ell D^\ell x = \sum_{\ell=0}^{k} b_\ell D^\ell y. \tag{3.4.3}$$

Suppose that H is a linear shift-invariant system such that $y = Hx$ if x and y satisfy a differential equation of this form. The response of H to the complex exponential signal e^{st} satisfies $He^{st} = \lambda H(s) e^{st}$. Substituting $x(t) = e^{st}$ and $y = \lambda H(s) e^{st}$ into the differential equation gives

$$\sum_{\ell=0}^{m} a_\ell s^\ell e^{st} = \sum_{\ell=0}^{k} b_\ell s^\ell \lambda H(s) e^{st} = \lambda H(s) \sum_{\ell=0}^{k} b_\ell s^\ell e^{st}.$$

Rearranging we find that the transfer function λH satisfies

$$\lambda H(s) = \frac{\sum_{\ell=0}^{m} a_\ell s^\ell e^{st}}{\sum_{\ell=0}^{k} b_\ell s^\ell e^{st}} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}. \tag{3.4.4}$$

The transfer function takes the form of a polynomial divided by a polynomial. The transfer function associated with a linear differential equation with constant coefficients (3.4.3) always takes this form. We will study these transfer functions in greater detail when we introduce the Laplace transform in Chapter 4.

3.5 The spectrum

It is of interest to focus on the transfer function when s is purely imaginary, that is, when $s = j\omega$ with $\omega \in \mathbb{R}$. In this case the complex exponential signal takes the form

$$e^{j\omega t} = e^{j2\pi f} = \cos(2\pi f t) + j\sin(2\pi f t)$$

where $\omega = 2\pi f$. This signal is oscillatory when $f \neq 0$ and does not decay or explode as $|t| \to \infty$. Let $H \in X \to Y$ be a linear shift-invariant system with domain X containing the signal $e^{j2\pi f t}$, that is, $j2\pi f t \in \operatorname{cep} X$. We denote by ΛH the function satisfying

$$\Lambda H(f) = \lambda H(j2\pi f) \qquad j2\pi f \in \operatorname{cep} X$$

called the **spectrum** of H. It will typically be the case that $\operatorname{cep} X$ contains the entire imaginary axis and so the domain of ΛH is the set of real numbers $\mathbb{R}$. In this case the spectrum is a signal, that is, $\lambda H \in \mathbb{R} \to \mathbb{C}$. The independent variable f typically represents frequency in Hertz.

It follows from (3.4.1) that the response of H to $e^{j2\pi f t} \in X$ satisfies

$$He^{j2\pi f t} = \lambda H(j2\pi f)e^{j2\pi f t} = \Lambda H(f)e^{j2\pi f t}$$

It is of interest to consider the **magnitude spectrum** $|\Lambda H(f)|$ and the **phase spectrum** $\angle \Lambda H(f)$ separately. The notation $\angle$ denotes the **argument** (or **phase**) of a complex number. We have,

$$\Lambda H(f) = |\Lambda H(f)|\, e^{j\angle \Lambda H(f)}$$

and correspondingly

$$He^{j2\pi f t} = |\Lambda H(f)|\, e^{j(2\pi f t + \angle \Lambda H(f))}.$$

Taking real and imaginary parts we obtain the pair of real valued solutions

$$\begin{aligned} H\cos(2\pi f t) &= |\Lambda H(f)| \cos\big(2\pi f t + \angle \Lambda H(f)\big), \\ H\sin(2\pi f t) &= |\Lambda H(f)| \sin\big(2\pi f t + \angle \Lambda H(f)\big). \end{aligned} \tag{3.5.1}$$

Consider again the active RC circuit with H the system mapping input voltage x to output voltage y. According to (3.4.2) the spectrum of H is

$$\Lambda H(f) = -\frac{1}{1 + 2\pi RCfj}. \tag{3.5.2}$$

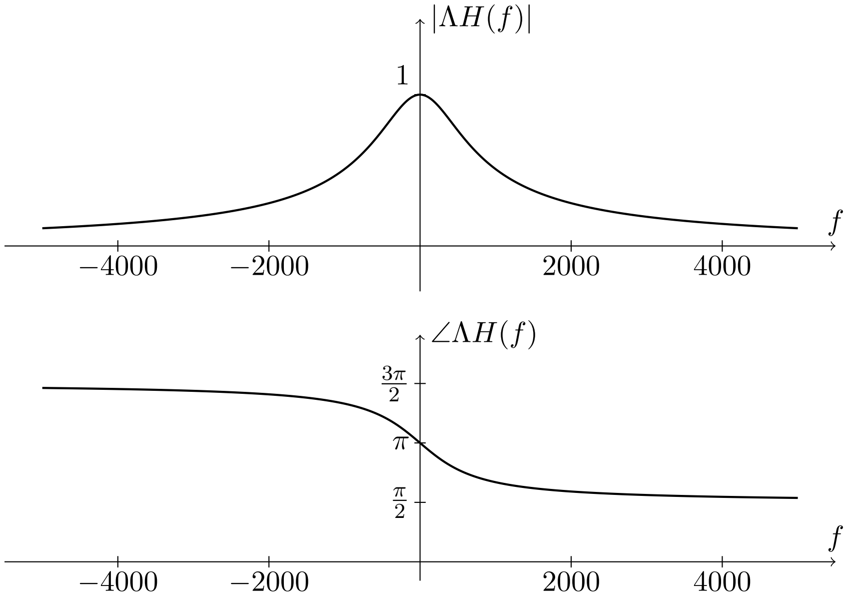

Figure 3.6: Magnitude spectrum (top) and phase spectrum (bottom) of the active RC circuit with $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$.

The magnitude and phase spectrum is

$$|\Lambda H(f)| = \big(1 + 4\pi^2 R^2 C^2 f^2\big)^{-\frac{1}{2}}, \qquad \angle \Lambda H(f) = \pi - \operatorname{atan}(2\pi R C f).$$

These are plotted in Figure 3.6 when $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$. Observe from the plot of the magnitude spectrum that a low frequency sinusoidal signal, say 100Hz or less, input to the active RC circuit results in a sinusoidal output signal with the same frequency and approximately the same amplitude. However, a high frequency sinusoidal signal, say greater than 1000Hz, input to the circuit results in a sinusoidal output signal with the same frequency, but smaller amplitude. For this reason RC circuits are called **low pass filters**.

Test 4 (**Spectrum of the active RC circuit**) We test the hypothesis that the active RC circuit satisfies (3.5.1). To do this sinusoidal signals at varying frequencies of the form

$$x_k(t) = \sin(2\pi f_k t), \qquad f_k = \left\lceil 110 \times 2^{k/2} \right\rfloor, \quad k = 0, 1, \dots, 12$$

are input to the active RC circuit constructed as in Test 3 with $R = R_1 = 27\text{k}\Omega$ and $C = C_2 = 10\text{nF}$. The notation $\lceil \cdot \rfloor$ denotes rounding to the nearest integer with half integers rounded up. In view of (3.5.1) the expected output signals are of the form

$$y_k(t) = |\Lambda H(f_k)| \sin\big(2\pi f_k t + \angle \Lambda H(f_k)\big), \qquad k = 0, 1, \dots, 12.$$

This equality can also be shown directly using the differential equation for the active RC circuit.

Using the soundcard each signal x_k is played for a period of approximately 1 second and approximately $F = 44100$ samples are obtained. On the soundcard hardware used for this test samples near the beginning and end of playback are distorted. This appears to be an unavoidable feature of the soundcard. To alleviate this we discard the first 10^4 samples and use only the $L = 8820$ samples that follow (corresponding to 200ms of signal). After this process we have samples $x_{k,1}, \dots, x_{k,L}$ and $y_{k,1}, \dots, y_{k,L}$ of the input and output signals corresponding with the kth signal x_k. The samples are expected to take the form

$$x_{k,\ell} \approx x_k(P\ell - \tau) = \rho \sin(2\pi f_k P\ell - \theta)$$

and

$$y_{k,\ell} \approx y_k(\ell P - \tau) = |\Lambda H(f_k)|\, \rho \sin\big(2\pi f_k P\ell - \theta + \angle \Lambda H(f_k)\big)$$

where $P = \frac{1}{F}$ is the sample period, the positive real number ρ corresponds with the gain on the input and output of the soundcard, and $\theta = 2\pi f_k \tau$ corresponds with delays caused by discarding the first 10^4 samples and also unavoidable delays that occur when starting soundcard playback and recording.

We will not measure the gain ρ nor the delay θ, but will be able to test the properties of the circuit without knowledge of these. To simplify notation put $\gamma = 2\pi f_k P$. From the samples of the input signal $x_{k,1}, \dots, x_{k,L}$ compute the complex number

$$A = \frac{2j}{L} \sum_{\ell=1}^{L} x_{k,\ell} e^{-j\gamma\ell} \approx \frac{2j}{L} \sum_{\ell=1}^{L} \rho \sin(\gamma\ell - \theta) e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where $\alpha = \rho e^{-j\theta}$ and α^* denotes the complex conjugate of α and

$$C = e^{-\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)} \qquad \text{(Exercise 3.8)}.$$

Similarly, from the samples of the output signal $y_{k,1}, \dots, y_{k,L}$ we compute the complex number

$$B = \frac{2j}{L} \sum_{\ell=1}^{L} y_{k,\ell} e^{-j\gamma\ell} \approx \beta + \beta^* C$$

where $\beta = \rho e^{-j\theta}\Lambda H(f_k) = \alpha\Lambda H(f_k)$. Now compute the quotient

$$Q_k = \frac{B - B^*C}{A - A^*C} \approx \frac{\beta(1 + |C|^2)}{\alpha(1 + |C|^2)} = \frac{\beta}{\alpha} = \Lambda H(f_k).$$

Thus, we expect the quotient Q_k to be close to the spectrum of the active RC circuit evaluated at frequency f_k. We will test this hypothesis by observing the magnitude and phase of Q_k individually, that is, we will test the expected relationships

$$|Q_k| \approx |\Lambda H(f_k)| = \sqrt{\frac{1}{1 + 4\pi^2 R^2 C^2 f_k^2}}$$

and

$$\angle Q_k \approx \angle \Lambda H(f_k) = \pi - \operatorname{atan}(2\pi RC f_k)$$

for each $k = 0, \dots, 12$. Figure 3.7 plots the hypothesised magnitude and phase spectrum alongside the measurements Q_k for $k = 0, \dots, 12$.

Exercises

3.1. State whether each of the following systems are: causal, linear, shift-invariant, or stable. Plot the impulse and step response of the systems whenever they exist. In each case, assume the domain to be the set of locally integrable signals.

(a) $Hx(t) = 3x(t-1) - 2x(t+1)$

(b) $Hx(t) = \sin\big(2\pi x(t)\big)$

(c) $Hx(t) = t^2 x(t)$

(d) $Hx(t) = \int_{-1/2}^{1/2} \cos(\pi\tau)x(t+\tau)d\tau$

3.2. Show that the system $Hx(t) = \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau$ is linear shift-invariant and regular. Find and sketch the impulse response and the step response.

3.3. Let h be a locally integrable signal. Show that the set $\operatorname{dom} h$ defined in Section 3.1 on page 33 is a linear shift-invariant space.

3.4. Show that $\operatorname{dom} u$ where u is the step function is the subset of locally integrable signals such that $\int_{-\infty}^{0} |x(t)|\, dt < \infty$.

3.5. Show that a regular system is stable if and only if its impulse response is absolutely integrable.

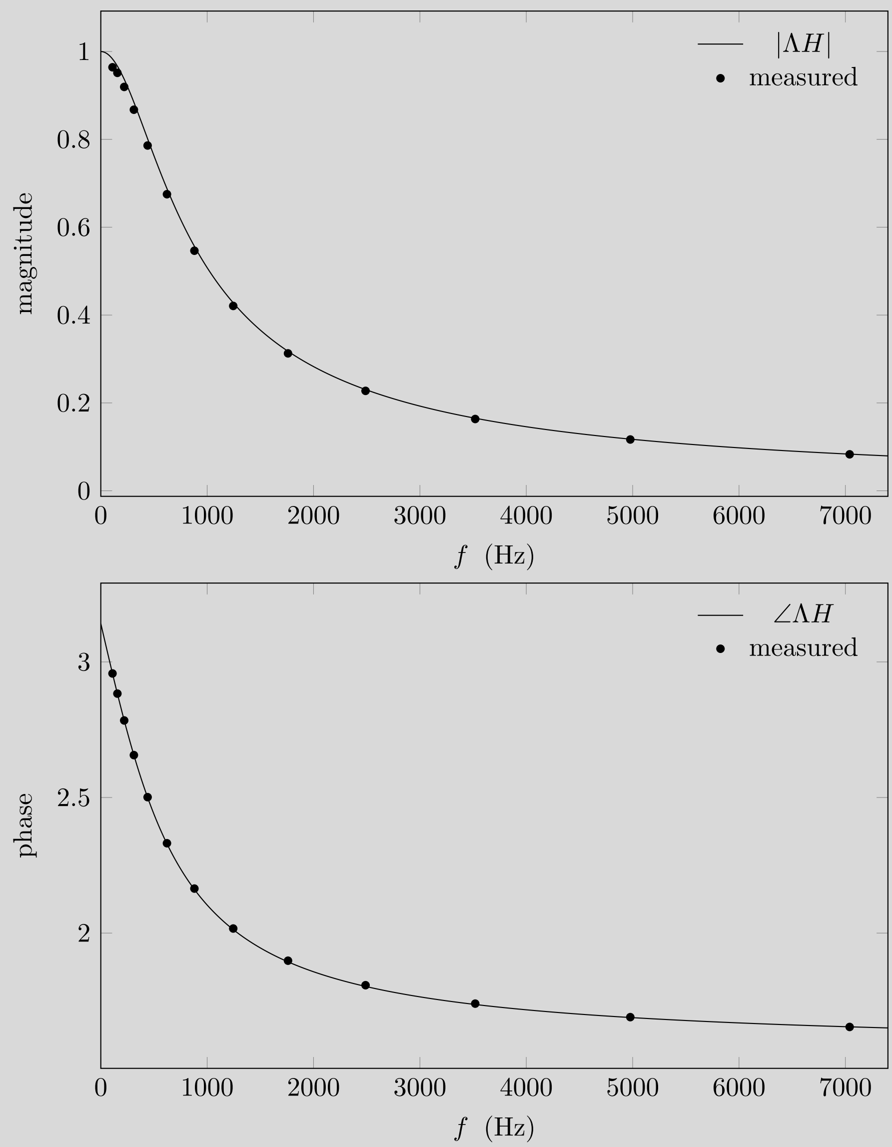

Figure 3.7: Hypothesised magnitude spectrum $|\Lambda H(f)|$ (top) and phase spectrum $\angle\Lambda H(f)$ (bottom) and the measured magnitude and phase spectrum $|Q_k|$ and $\angle Q_k$ for $k = 0, \ldots, 12$ (dots).

3.6. Define signals $x(t) = u(t)$, $y(t) = u(-t)$, and $z(t) = \Pi(t) - \Pi(t-1)$ where u is the step function and Π is the rectangular pulse. Plot x, y, and z and show that the associative property of convolution does not hold for these signals. That is, show that $x * (y * z) \neq (x * y) * z$.

3.7. Show that $\sum_{\ell=1}^{L} e^{\beta\ell} = \frac{e^{\beta(L+1)} - e^{\beta}}{e^{\beta} - 1}$ (Hint: sum a geometric progression).

3.8. Show that

$$\frac{2j}{L} \sum_{\ell=1}^{L} \sin(\gamma\ell - \theta) e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where $\alpha = e^{-j\theta}$ and $C = e^{-j\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)}$. (Hint: solve Exercise 3.7 first and then use the formula $2j \sin(x) = e^{jx} - e^{-jx}$).

∗3.9. Show that the convolution of two absolutely integrable signals is absolutely integrable.

Chapter 4

The Laplace transform

In the previous chapter we studied the properties of linear shift-invariant systems. We considered regular systems for which the response to input signal x is given by $h * x$, that is, by the convolution of x with the impulse response h of the system. We discovered that complex exponential signals of the form e^{st} were eigenfunctions of linear shift-invariant systems, that is, if H is the linear-shift-invariant system with domain X, then the response of H to input signal $e^{st} \in X$ satisfies

$$He^{st} = \lambda H(s)e^{st}$$

where $\lambda H(s)$ is a complex number that depends on $s \in \mathbb{C}$, but not on $t \in \mathbb{R}$. Considered as a function of s, the expression λH is called the transfer function of the system H. The domain of λH is the set of complex numbers s such that $e^{st} \in X$ and this set is denoted by $\operatorname{cep} X$.

Now consider the special case where H is a regular system with impulse response h and domain $\operatorname{dom} h$. The response of H to a complex exponential signal $e^{st} \in \operatorname{dom} h$ is given by convolution of h with e^{st}, that is,

$$He^{st} = h * e^{st} = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

It follows that the transfer function of the regular system H satisfies

$$\lambda H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \qquad s \in \operatorname{cep}\operatorname{dom} h.$$

This is also know as the **Laplace transform** of the signal h. The set $\operatorname{cep}\operatorname{dom} h$ is called the **region of convergence** of the Laplace transform of h. We will denote the region of convergence by the simpler notation $\operatorname{roc} h = \operatorname{cep}\operatorname{dom} h$ in what follows.

Let x be a signal. The Laplace transform of x is denoted by $\mathcal{L}x$. It is the complex valued function of $\operatorname{roc} x$ satisfying

$$\mathcal{L}x(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad s \in \operatorname{roc} x. \tag{4.0.1}$$

The domain of $\mathcal{L}x$ is the region of convergence $\operatorname{roc} x = \operatorname{cep}\operatorname{dom} x$.

4.1 Regions of convergence

We now study the Laplace transform and the region of convergence in more detail. The region of convergence $\operatorname{roc} x = \operatorname{cep}\operatorname{dom} x$ is precisely the set of complex numbers s such that $x(t)e^{-st}$ is absolutely integrable (Exercise 4.19), that is,

$$\int_{-\infty}^{\infty} \left|x(t)e^{-st}\right| dt < \infty \qquad \text{if and only if } s \in \operatorname{roc} x.$$

For example, consider the right sided signal $e^{\alpha t}u(t)$. We have

$$\int_{-\infty}^{\infty} \left|e^{\alpha t}u(t)e^{-st}\right| dt = \int_{0}^{\infty} e^{\operatorname{Re}(\alpha - s)t} dt = \lim_{t\to\infty} \frac{e^{\operatorname{Re}(\alpha-s)t}}{\operatorname{Re}(\alpha - s)} - \frac{1}{\operatorname{Re}(\alpha - s)}.$$

The limit above converges if and only if $\operatorname{Re}(\alpha - s) < 0$ and so the region of convergence of the Laplace transform of $e^{\alpha t}u(t)$ is

$$\operatorname{roc} e^{\alpha t}u(t) = \{s \in \mathbb{C}\ ;\ \operatorname{Re} s > \operatorname{Re} \alpha\}.$$

Figure 4.1 shows the region of convergence when $\operatorname{Re}\alpha = -2$. Applying (4.0.1) we find that

$$\mathcal{L}(e^{\alpha t}u(t)) = \int_{-\infty}^{\infty} e^{\alpha t}e^{-st}u(t)dt = \lim_{t\to\infty} \frac{e^{(\alpha-s)t}}{\alpha - s} - \frac{1}{\alpha - s}.$$

The limit converges to zero when $\operatorname{Re}(\alpha - s) < 0$, that is, when $s \in \operatorname{roc} e^{\alpha t}u(t)$, and so the Laplace transform is

$$\mathcal{L}(e^{\alpha t}u(t)) = \frac{1}{s - \alpha} \qquad \operatorname{Re} s > \operatorname{Re} \alpha.$$

Now consider the left sided signal $e^{\beta t}u(-t)$. The region of convergence is

$$\operatorname{roc} e^{\beta t}u(-t) = \{s \in \mathbb{C}\ ;\ \operatorname{Re} s < \operatorname{Re} \beta\}$$

and the Laplace transform is

$$\mathcal{L}(e^{\beta t}u(-t)) = \lim_{t\to-\infty} \frac{e^{(\beta-s)t}}{\beta - s} + \frac{1}{\beta - s} = \frac{1}{\beta - s} \qquad \operatorname{Re} s < \operatorname{Re} \beta.$$

The signal $ae^{\alpha t}u(t) + be^{\beta t}u(-t)$ has Laplace transform

$$\begin{aligned}\mathcal{L}\big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big) &= \int_{-\infty}^{\infty} \big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big)e^{-st}dt \\ &= a\mathcal{L}(e^{\alpha t}u(t)) + b\mathcal{L}(e^{\beta t}u(-t))\end{aligned}$$

with region of convergence

$$\operatorname{roc}(ae^{\alpha t}u(t) + be^{\beta t}u(-t)) = \{s \in \mathbb{C}\,;\ \operatorname{Re}\alpha < \operatorname{Re}s < \operatorname{Re}\beta\}.$$

This region is shown in Figure 4.1 when $\operatorname{Re}\alpha = -2$ and $\operatorname{Re}\beta = 3$. In the previous equation we have discovered that the Laplace transform is **linear**, that is, for signals x and y and non zero complex numbers a and b, the Laplace transform of the linear combination $ax + by$ is

$$\mathcal{L}(ax + by) = a\mathcal{L}x + b\mathcal{L}y \qquad s \in \operatorname{roc}(ax + by) = \operatorname{roc}x \cap \operatorname{roc}y. \tag{4.1.1}$$

The region of convergence is the intersection of the regions of convergence of $\mathcal{L}x$ and $\mathcal{L}y$.

In the previous example the region of convergence is the empty set $\emptyset$ if $\operatorname{Re}\alpha \geq \operatorname{Re}\beta$. The Laplace transform is said not to exist in this case. Other signals have this property. For example, the signal $x(t) = 1$ has no Laplace transform because

$$\int_{-\infty}^{\infty} \left|e^{-st}\right| dt = \lim_{t\to-\infty} \frac{e^{-\operatorname{Re}st}}{s} - \lim_{t\to\infty} \frac{e^{-\operatorname{Re}st}}{s}$$

and the limit as $t \to -\infty$ converges only when $\operatorname{Re}s < 0$ while the limit as $t \to \infty$ converges only when $\operatorname{Re}s > 0$.

As a final example, consider the rectangular pulse

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The region of convergence is the entire complex plane, $\operatorname{roc}\Pi = \mathbb{C}$, because

$$\int_{-\infty}^{\infty} \left|\Pi(t)e^{-st}\right| dt = \int_{-1/2}^{1/2} e^{-\operatorname{Re}st} dt = \frac{e^{\operatorname{Re}s/2} - e^{-\operatorname{Re}s/2}}{s} < \infty$$

for all $s \in \mathbb{C}$. At $s = 0$ the above expression is given the value 1 by appealling to continuity (Exercise 4.16). The Laplace transform of Π is

$$\mathcal{L}\Pi = \int_{-\infty}^{\infty} \Pi(t)e^{-st} dt = \int_{-1/2}^{1/2} e^{-st} dt = \frac{e^{s/2} - e^{-s/2}}{s} \qquad s \in \mathbb{C}. \tag{4.1.2}$$

The examples just given exhibit all the possible types of regions of convergence. The region of convergence is either the entire complex plane, a left or right half plane, a vertical strip, or the empty set.

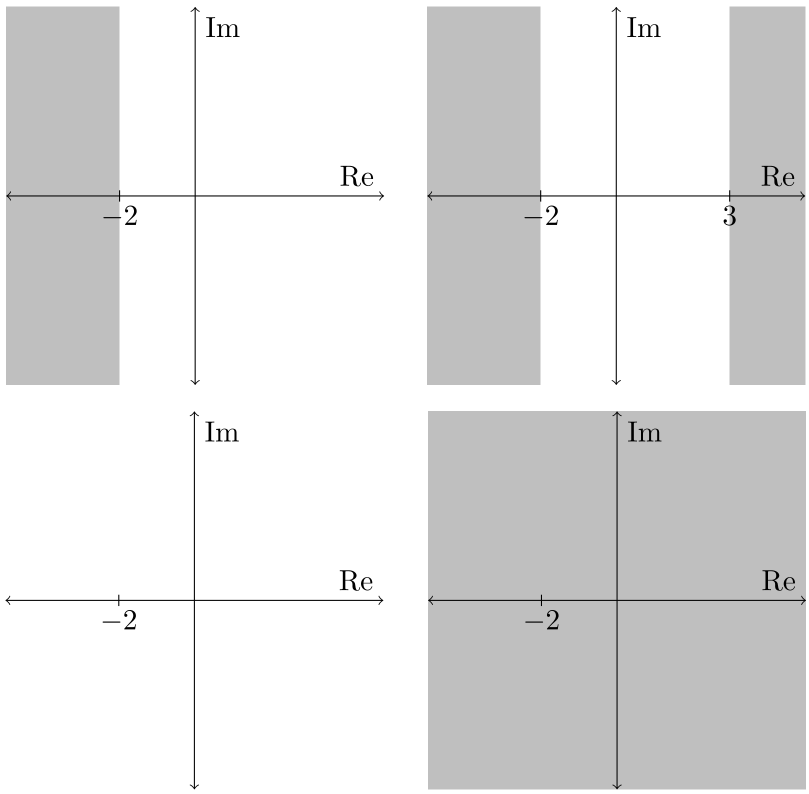

Figure 4.1: Regions of convergence (unshaded) of the Laplace transforms of the signal $e^{-2t}u(t)$ (top left), the signal $e^{-2t}u(t) + e^{3t}u(-t)$ (top right), the rectangular pulse Π (bottom left), and the constant signal $x(t) = 1$ (bottom right).

4.2 The inverse Laplace transform

Given the Laplace transform $\mathcal{L}x \in \operatorname{roc} x \to \mathbb{C}$ the signal x can be recovered by the **inverse Laplace transform**

$$x(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma - j\omega} \mathcal{L}x(s) e^{st} ds,$$

where σ is any real number inside the region of convergence $\operatorname{roc} x$. Solving the integral above typically requires a special type of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following Laplace transform pair

$$\mathcal{L}\big(t^n u(t)\big) = \frac{n!}{s^{n+1}} \qquad \operatorname{Re} s > 0, \tag{4.2.1}$$

where $n \geq 0$ is an integer (Exercise 4.2). The Laplace transforms of the signal $x(t)$ and the signal $e^{\alpha t} x(t)$ are related,

$$\begin{aligned} \mathcal{L}\big(e^{\alpha t} x(t)\big)(s) &= \int_{-\infty}^{\infty} e^{\alpha t} x(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-(s-\alpha)t} dt \\ &= \mathcal{L}x(s - \alpha) \qquad s - \alpha \in \operatorname{roc} x. \end{aligned} \tag{4.2.2}$$

The region of convergence of $\mathcal{L}(e^{\alpha t} x(t))$ is all those complex numbers s such that $s - \alpha \in \operatorname{roc} x$. This is called the **frequency-shift rule**. Combining the frequency-shift rule with (4.2.1) we obtain the transform pair

$$\mathcal{L}\big(t^n e^{\alpha t} u(t)\big) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \operatorname{Re} s > \operatorname{Re} \alpha, \tag{4.2.3}$$

where $n \geq 0$ is an integer. This is the only Laplace transform pair we require here.

A useful relationship exists between the Laplace transform of a signal $x(t)$ and its time-scaled version $x(\alpha t)$ where $\alpha \neq 0$,

$$\mathcal{L}\big(x(\alpha t)\big)(s) = \frac{1}{|\alpha|} \mathcal{L}x(s/\alpha), \qquad \operatorname{Re}(s/\alpha) \in \operatorname{roc} x. \tag{4.2.4}$$

The region of convergence of $\mathcal{L}\big(x(\alpha t)\big)$ is those complex numbers s such that $s/\alpha \in \operatorname{roc} x$. This is called the **time-scaling property** of the Laplace transform (Excercise 4.12).

4.3 The transfer function and the Laplace transform

We have already discovered that the transfer function λH of a regular system H with impulse response h is equal to the Laplace transform $\mathcal{L}h$ of the impulse response, that is,

$$\lambda H(s) = \mathcal{L}h(s) \qquad s \in \operatorname{roc} h.$$

The transfer functions of the shifter T_τ and differentiator D can be obtained by inspection. For the shifter

$$T_\tau e^{st} = e^{s(t-\tau)} = e^{-s\tau}e^{st} \qquad \text{and so} \qquad \lambda T_\tau(s) = e^{-s\tau}. \tag{4.3.1}$$

For the special case of the identity system T_0 we obtain $\lambda T_0 = 1$. For the differentiator

$$De^{st} = \frac{d}{dt}e^{st} = se^{st} \qquad \text{and so} \qquad \lambda D(s) = s.$$

More generally, for the kth differentiator

$$D^k e^{st} = \frac{d^k}{dt^k}e^{st} = s^k e^{st} \qquad \text{and so} \qquad \lambda D^k(s) = s^k. \tag{4.3.2}$$

The domain of λT_τ and of λD^k is the entire complex plane $\mathbb{C}$. These results motivate assigning the following Laplace transforms to the delta "function" δ, its shift $T_\tau\delta = \delta(t-\tau)$, and the δ^k symbol,

$$\mathcal{L}\delta = 1, \qquad \mathcal{L}\big(\delta(t-\tau)\big) = e^{-s\tau}, \qquad \mathcal{L}\delta^k = s^k.$$

These conventions are common in the literature [Oppenheim et al., 1996].

We now study the transfer function of a system formed by composition. Let $F \in X \to Y$ and $G \in Y \to Z$ be linear shift-invariant systems with domains X and Y and suppose that the complex exponential signal $e^{st} \in X$ if $e^{st} \in Y$, that is, $\operatorname{cep} X \subset \operatorname{cep} Y$. This will be the only case of interest to us. Let $H \in X \to Z$ be the system formed by the composition $H = GF$. The response of H to the signal $e^{st} \in X$ satisfies

$$He^{st} = GFe^{st} = G\big(\lambda F(s)e^{st}\big) = \lambda F(s)Ge^{st}$$

and since e^{st} is also in Y,

$$He^{st} = \lambda F(s)\lambda G(s)e^{st} = \lambda H(s)e^{st}.$$

It follows that

$$\lambda H(s) = \lambda F(s)\lambda G(s) \qquad s \in \operatorname{cep} X = \operatorname{cep} X \cap \operatorname{cep} Y. \tag{4.3.3}$$

That is, the transfer function of a composition of linear shift-invariant systems is the multiplication of the transfer functions of those systems.

Now suppose that F and G are regular systems with impulse responses f and g. We showed in Section 3.3 that the composition $H = GF$ is a regular system with impulse response given by the convolution $h = g * f$ and we take its domain to be $\operatorname{dom} f\ g$. It can be shown that $e^{st} \in \operatorname{dom} f\ g$ if and only if $s \in \operatorname{roc} f \cap \operatorname{roc} g$, that is, $\operatorname{cep} \operatorname{dom} f\ g = \operatorname{roc} f \cap \operatorname{roc} g$ (Exercise 4.20). Thus, for $s \in \operatorname{roc} f \cap \operatorname{roc} g$ we have

$$\lambda H(s) = \mathcal{L}h(s) \qquad \lambda F(s) = \mathcal{L}f(s) \qquad \lambda G(s) = \mathcal{L}g(s),$$

and using (4.3.3) and dropping the "(s)"'s for notationial clarity, we obtain,

$$\mathcal{L}(f * g) = \mathcal{L}h = \lambda H = \lambda F \lambda G = \mathcal{L}f\mathcal{L}g$$

for $s \in \operatorname{roc} f \cap \operatorname{roc} g$. Putting $x = f$ and $y = g$, we obtain the **convolution theorem**,

$$\mathcal{L}(x * y) = \mathcal{L}x\mathcal{L}y \tag{4.3.4}$$

with region of convergence $\operatorname{roc} x \cap \operatorname{roc} y$. In words: the Laplace transform of a convolution of signals is the multiplication of their Laplace transforms. The region of convergence of $\mathcal{L}(x * y)$ is the intersection of the regions of convergence of $\mathcal{L}x$ and $\mathcal{L}y$, that is, $\operatorname{roc} x \cap \operatorname{roc} y$.

Let H be a regular system with impulse response h and let $y = Hx$ be the response of the system H to input signal $x \in \operatorname{dom} h$. We have $y = h * x$ and the convolution theorem asserts that

$$\mathcal{L}y = \mathcal{L}h\mathcal{L}x = \lambda H \mathcal{L}x \tag{4.3.5}$$

with region of convergence $\operatorname{roc} h \cap \operatorname{roc} x$. Thus, the Laplace transform of the response $y = Hx$ is the transfer function of the regular system H multiplied by the Laplace transform of the input signal x. This result also holds for the shifter, that is,

$$\mathcal{L}T_\tau x = \lambda T_\tau \mathcal{L}x = e^{-s\tau}\mathcal{L}x$$

with region of convergence $\operatorname{roc} x$. This is called the **time-shift property** of the Laplace transform (Exercise 4.4). The result also holds for the differentiator, that is,

$$\mathcal{L}Dx = \lambda D \mathcal{L}x = s\mathcal{L}x$$

with region of convergence $\operatorname{roc} x$ under the added assumptions that x is differentiable, i.e. $x \in C^1$, and that

$$\lim_{t\to\infty} x(t)e^{-st} = \lim_{t\to-\infty} x(t)e^{-st} = 0 \qquad \text{when } s \in \operatorname{roc} x.$$

This is called the **differentiation property** of the Laplace transform (Exercise 4.5). Observe that the added assumptions are true if, for example, $x(t)$ is finite.

In Chapter 2 we modelled electrical, mechanical, and electromechanical devices by linear differential equations with constant coefficients of the form

$$\sum_{\ell=0}^{m} a_\ell D^\ell x = \sum_{\ell=0}^{k} b_\ell D^\ell y.$$

Suppose that H is a linear shift-invariant system such that $y = Hx$ if x and y satisfy a differential equation of this form. In Section 3.4 we found that the transfer function of H is a ratio of polynomials, that is,

$$\lambda H(s) = \frac{\sum_{\ell=0}^{m} a_\ell s^\ell e^{st}}{\sum_{\ell=0}^{k} b_\ell s^\ell e^{st}} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Properties of H can be obtained by inspecting this transfer function. For example, the impulse response of H (if it exists) can be obtained by applying the inverse Laplace transform.

We now apply these results to the differential equations that model the RC electrical circuit from Figure 2.1 and the mass spring damper from Figure 2.2. The RC circuit is an example of what is called a **first order system** and the mass spring damper is an example of what is called a **second order system**.

4.4 First order systems

Recall the passive electrical RC circuit from Figure 2.1. The differential equation modelling this circuit is (2.0.1),

$$x = y + RCDy$$

where x is the input voltage signal, y is the voltage over the capacitor, and R and C are the resistance and capacitance. The RC circuit is an example of a **first order system** so called because the highest order derivative that occurs is of order one, that is, D^k with $k = 1$. Let H be a system mapping the input voltage signal x to the output voltage signal y. From (3.4.4) the transfer function λH is found to satisfy

$$\lambda H(s) = \frac{1}{1 + RCs} = \frac{r}{r + s}$$

where $r = \frac{1}{RC}$. The value $\frac{1}{r} = RC$ is called the **time constant**. The impulse response of H is given by the inverse of this Laplace transform. There are two signals with Laplace transform $\frac{r}{r+s}$: the right sided signal $re^{-rt}u(t)$ with region of convergence $\operatorname{Re} s > -r$, and the left sided signal $-re^{-rt}u(-t)$ with region of convergence $\operatorname{Re} s < -r$. The RC circuit (and in fact all physically realisable systems) are expected to be causal. For this reason, the left sided signal $-re^{-rt}u(-t)$ cannot be the impulse response of H. The impulse response is the right sided signal

$$h(t) = re^{-rt}u(t).$$

Given an input voltage signal x we can now find the corresponding output signal $y = Hx$ by convolving x with the impulse response h. That is,

$$y = Hx = h * x = \int_{-\infty}^{\infty} re^{-r\tau}u(\tau)x(t-\tau)d\tau = r\int_{0}^{\infty} e^{-r\tau}x(t-\tau)d\tau.$$

If $r \geq 0$ the impulse response is absolutely integrable, that is,

$$\begin{aligned} \|h\|_1 &= \int_{-\infty}^{\infty} \left|re^{-rt}u(t)\right| dt \\ &= r\int_{0}^{\infty} e^{-rt}dt \\ &= 1 - \lim_{t\to\infty} e^{-rt} = 1, \end{aligned}$$

and the system is stable (Exercise 3.5). However, if $r < 0$ the impulse response is not absolutely integrable and the system is not stable. Figure 4.3 shows the impulse response when $r = -\frac{1}{5}, -\frac{1}{3}, -\frac{1}{2}, 1, 2$. In a passive electrical RC circuit the resistance R and capacitance C are always positive and $r = \frac{1}{RC}$ is positive. For this reason, passive electrical RC circuits are always stable.

From (3.1.3), the step response Hu is given by applying the integrator I_∞ to the impulse response, that is,

$$Hu = I_\infty h = \int_{-\infty}^{t} re^{-r\tau}u(\tau)d\tau = \begin{cases} r\int_0^t e^{-r\tau}d\tau & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

or more simply

$$Hu = \big(1 - e^{-rt}\big)u(t). \tag{4.4.1}$$

This step response in plotted in Figure 4.3.

Test 5 (**The impulse response of the active RC circuit**) In this test we again use the active RC circuit from Test 3 with resistors $R = R_1 = R_2 = 27\text{k}\Omega$ and capacitor $C = C_2 = 10\text{nF}$. In Test 3 we applied the differential equation (2.2.4) to the reconstructed output signal $\tilde{y}$ and asserted that the resulting signal was close to the reconstructed input signal $\tilde{x}$. In this test we instead convolve the input signal $\tilde{x}$ with the impulse response

$$h(t) = -\tfrac{1}{RC}e^{-t/RC}u(t) = -re^{-rt}u(t), \qquad r = \tfrac{1}{RC} = \frac{10^5}{27}$$

and assert that the resulting signal is close to the output signal $\tilde{y}$. That is, we test the expected relationship

$$\tilde{y} \approx h * \tilde{x} = \int_{-\infty}^{\infty} h(\tau)\tilde{x}(t-\tau)d\tau.$$

From (1.3.4),

$$\begin{aligned}\tilde{y}(t) &\approx \int_{-\infty}^{\infty} h(\tau) \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - F\tau - \ell)d\tau \\ &= \sum_{\ell=1}^{L} x_\ell \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(Ft - F\tau - \ell)d\tau \\ &= \sum_{\ell=1}^{L} x_\ell g(Ft - \ell)\end{aligned}$$

where the function

$$g(t) = \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(t - F\tau)d\tau = -r \int_{0}^{\infty} e^{-r\tau} \operatorname{sinc}(t - F\tau)d\tau.$$

An approximation of $g(t)$ is made by the trapezoidal sum

$$g(t) \approx \frac{K}{2N}\left(p(0) + p(K) + 2\sum_{n=1}^{N-1} p(\Delta n)\right),$$

where $p(\tau) = h(\tau)\operatorname{sinc}(t - F\tau)$ and

$$K = -RC \log\left(10^{-3}\right), \qquad N = \lceil 10FK \rceil, \qquad \Delta = K/N.$$

Figure 4.2 plots the input signal $\tilde{x}$, output signal $\tilde{y}$, and hypothesised output signal $h * \tilde{x}$ over a 4ms window.

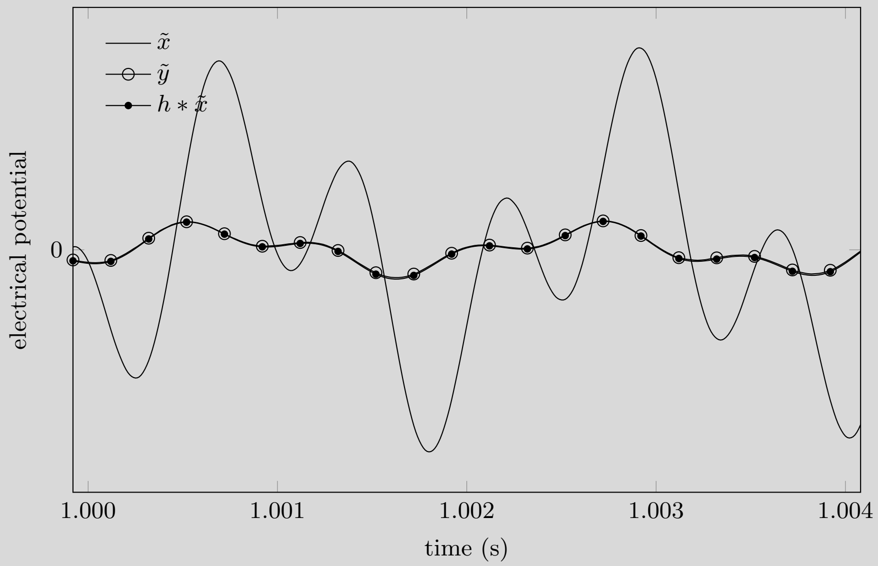

Figure 4.2: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle), and hypothesised output signal $h * \tilde{x}$ (solid line with dot).

4.5 Second order systems

Consider the mass spring damper system from Figure 2.2 that is described by the equation

$$f = Kp + BDp + MD^2p \tag{4.5.1}$$

where f is the force applied to the mass M and p is the position of the mass and K and B are the spring and damping coefficients. The mass spring damper is an example of a **second order system** because it contains differeniators D^2 of order at most two. Another example of a second order system is the Sallen-Key active electrical circuit depicted in Figure 2.10. In Section 2 we were able to find the force f corresponding with a given position signal p. Suppose that H is a linear shift-invariant system mapping f to p, that is, such that $p = Hf$. We will find the impulse response of H. From 3.4.4 the transfer function is found to satisfy

$$\lambda H(s) = \frac{1}{K + Bs + Ms^2}.$$

We can invert this Laplace transform to obtain the impulse response. There are three cases to consider depending on whether the quadratic $K+Bs+Ms^2$ has two distinct real roots, is irreducible (does not have real roots), or has two identical real roots.

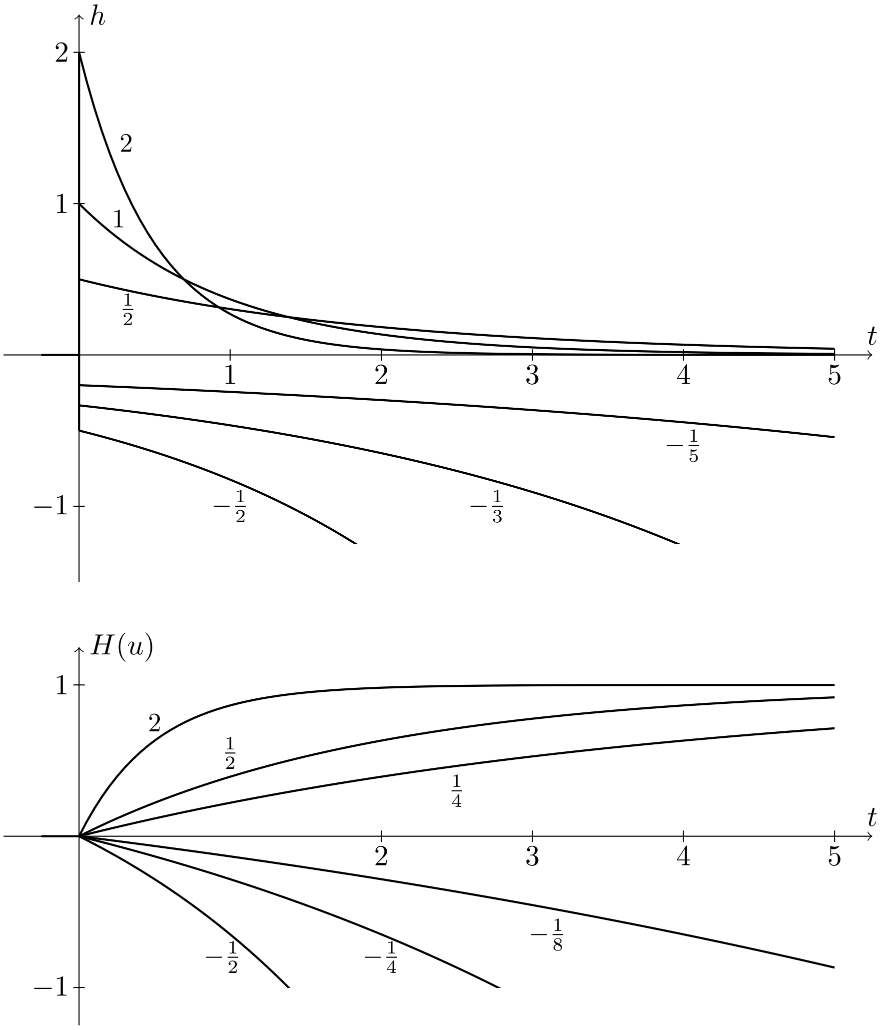

Figure 4.3: Top: impulse response of a first order system with $r = -\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{5}$, $\frac{1}{2}$, 1, 2. Bottom: step response of a first order system with $r = -\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 2.

Case 1: (Distinct real roots) In this case, the roots are

$$\beta - \alpha, \qquad -\beta - \alpha,$$

where

$$\alpha = \frac{B}{2M}, \qquad \beta = \frac{\sqrt{B^2 - 4KM}}{2M}$$

and $B^2 - 4KM > 0$. By a partial fraction expansion (Exercise 4.9),

$$\begin{aligned} \lambda H(s) &= \frac{1}{M(s-\beta+\alpha)(s+\beta+\alpha)} \\ &= \frac{1}{2\beta M}\left(\frac{1}{s-\beta+\alpha} - \frac{1}{s+\beta+\alpha}\right). \end{aligned}$$

From (4.2.3) we obtain the transform pairs

$$\mathcal{L}(e^{(\beta-\alpha)t}u(t)) = \frac{1}{s-\beta+\alpha}, \qquad \mathcal{L}(e^{-(\beta+\alpha)t}u(t)) = \frac{1}{s+\beta+\alpha}.$$

As in Section 4.4, other signals with these Laplace transforms are discarded because they do not lead to an impulse response that is zero for $t < 0$. That is, they do not lead to a causal system H. The impulse response of H is thus

$$h(t) = \frac{1}{2\beta M}u(t)e^{-\alpha t}\big(e^{\beta t} - e^{-\beta t}\big).$$

This is a sum of the impulse responses of two first order systems.

Case 2: (Distinct imaginary roots) The solution is as in the previous case, but now $4KM - B^2 > 0$ and β is imaginary. Put $\theta = \beta/j$ so that

$$e^{\beta t} - e^{-\beta t} = e^{j\theta t} - e^{-j\theta t} = 2j\sin(\theta t).$$

The impulse response of H is

$$h(t) = \frac{1}{\theta M}u(t)e^{-\alpha t}\sin(\theta t).$$

Case 3: (Identical roots) In this case, the two roots are equal to $-\alpha$ and

$$\lambda H(s) = \frac{1}{M(s+\alpha)^2}.$$

From (4.2.3) we obtain the transform pair

$$\mathcal{L}\big(te^{-\alpha t}u(t)\big) = \frac{1}{(s+\alpha)^2}$$

and this is the only signal with this Laplace transform that leads to a causal impulse response. The impulse response of H is thus

$$h(t) = \frac{1}{M} t e^{-\alpha t} u(t).$$

A second order system is called **overdamped** when there are two distinct real roots, **underdamped** when their are two distinct imaginary roots, and **critically damped** when the roots are identical. The different types of impulse responses for are plotted in Figure 4.4.

With no damping (i.e. damping coefficient $B = 0$) the roots are of the form $\pm\beta$ and have no real part. In this case, the impulse response is

$$h(t) = \frac{1}{\theta M} u(t) \sin(\theta t),$$

where $\theta = \beta/j = \sqrt{KM}$ is called the **natural frequency** of the second order system. This impulse response oscillates for all $t > 0$ without decay or explosion. Two identical roots occur when the damping coefficient $B = \sqrt{4KM}$ and this is sometimes called the **critical damping coefficient**.

The impulse response of a second order system is absolutely integrable when $\alpha = \frac{B}{2M} > 0$, but not when $\alpha \leq 0$. Thus, the system is stable when $\alpha > 0$ and not stable when $\alpha \leq 0$. For the mass spring damper both the mass M and damping coefficient B are positive and so mass spring dampers are always stable.

From (3.1.3) the step response $H(u)$ is given by applying the integrator I_∞ to the impulse response. There are three cases to consider depending on whether the system is overdamped, underdamped, or critically damped. When the system is overdamped the step response is

$$\begin{aligned} Hu = I_\infty h &= \frac{1}{2\beta M} \int_{-\infty}^{t} e^{-\alpha\tau} \big(e^{\beta\tau} - e^{-\beta\tau}\big) u(\tau) d\tau \\ &= \frac{1}{2\beta M} \int_{0}^{t} e^{-\alpha\tau} \big(e^{\beta\tau} - e^{-\beta\tau}\big) d\tau \\ &= \frac{1}{2\beta M} u(t) \left(\frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha} + \frac{e^{-(\beta+\alpha)t} - 1}{\beta + \alpha} \right). \end{aligned}$$

When the system is underdamped the step response is

$$\begin{aligned} Hu = I_\infty h &= \frac{1}{\theta M} \int_{0}^{t} e^{-\alpha\tau} \sin(\theta\tau) dt \\ &= u(t) \left(\frac{\theta - e^{-t\alpha}\big(\theta\cos(t\theta) + \alpha\sin(t\theta)\big)}{M\theta(\alpha^2 + \theta^2)} \right). \end{aligned}$$

When the system is critically damped the step response is

$$
\begin{aligned}
Hu = I_\infty h &= \frac{1}{\theta M}\int_0^t \frac{1}{M} t e^{-\alpha t} dt \\
&= \frac{1}{M\alpha^2} u(t)\big(1 - e^{-t\alpha s}(1 + t\alpha)\big).
\end{aligned}
$$

These step responses are plotted in Figure 4.5.

4.6 Poles, zeros, and stability

The transfer function of a system described by a linear differential equation with constant coefficients is of the form of (3.4.4), that is,

$$\lambda H(s) = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Factorising the polynomials on the numerator and denominator we obtain

$$\lambda H(s) = C\frac{(s-\alpha_0)(s-\alpha_1)\cdots(s-\alpha_m)}{(s-\beta_0)(s-\beta_1)\cdots(s-\beta_k)},$$

where $\alpha_0, \dots, \alpha_m$ are the roots of the numerator polynomial $a_0 + a_1 s + \cdots + a_m s^m$, and $\beta_0, \dots, \beta_k$ are the roots of the denominator polynomial $b_0 + b_1 s + \cdots + b_k s^k$, and $C = \frac{a_m}{b_m}$. That such a factorisation is always possible is called the **fundamental theorem of algebra** [Fine and Rosenberger, 1997]. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\lambda H(s) = C\frac{(s-\alpha_d)(s-\alpha_{d+1})\cdots(s-\alpha_m)}{(s-\beta_d)(s-\beta_{d+1})\cdots(s-\beta_k)}, \qquad (4.6.1)$$

where d is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \quad \alpha_1 = \beta_1, \quad \dots, \quad \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator $\alpha_d, \dots, \alpha_m$ are called the **zeros** and the roots from the denominator $\beta_d, \dots, \beta_m$ are called the **poles**. A **pole-zero plot** is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero. Pole-zero plots for the first order system from Section 4.4, the second order system from Section 4.5, and the system describing the PID controller (2.2.7) are shown in Figure 4.6.

It is always possible to apply partial fractions and write (4.6.1) in the form

$$\lambda H(s) = p(s) + \sum_{\ell \in K} \frac{A_\ell}{(s-\beta_\ell)^{r_\ell}},$$

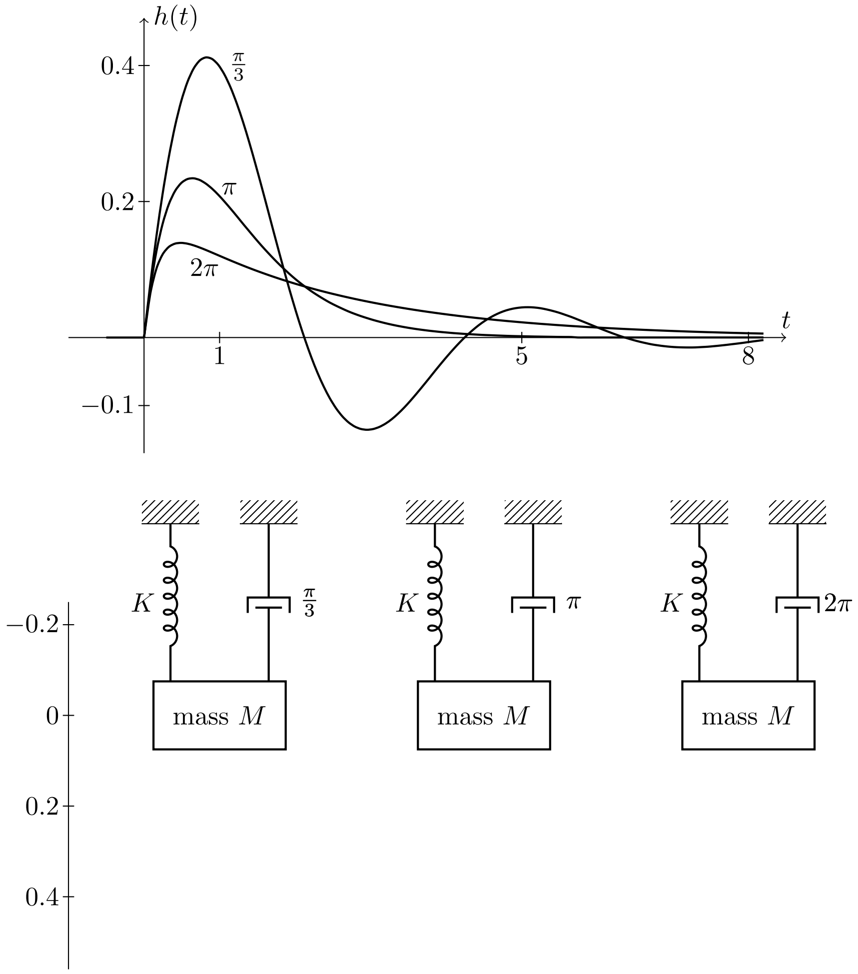

Figure 4.4: Impulse response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

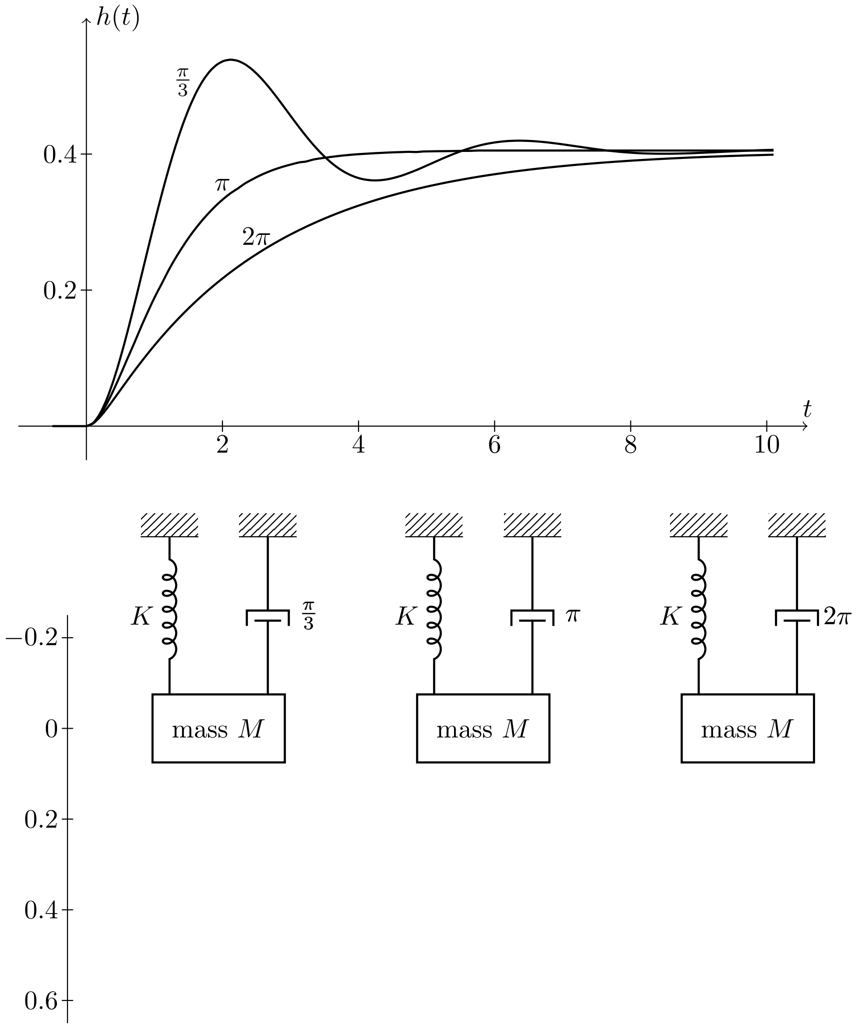

Figure 4.5: Step response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

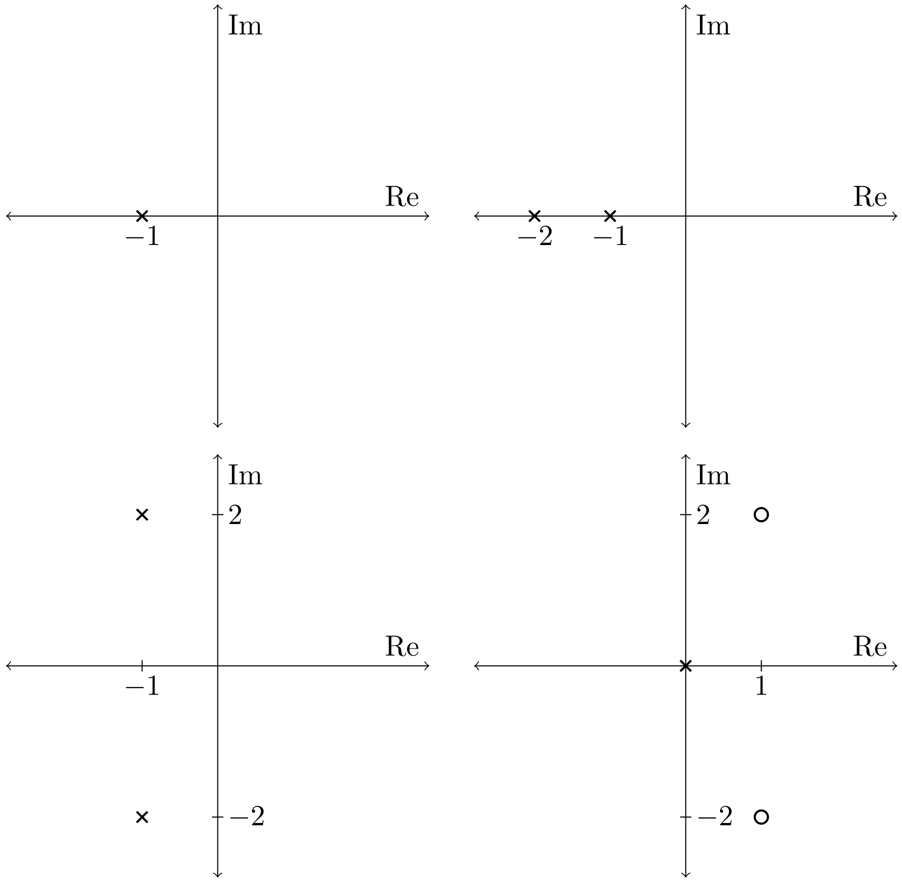

Figure 4.6: Top left: pole zero plot for the first order system $x = y + Dy$. There is a single pole at -1. Top right: pole zero plot for the overdamped second order system $x = 2y + 3Dy + D^2y$ that has two real poles at -1 and -2. Bottom left: pole zero plot for the underdamped second order system $x = 5y + 2Dy + D^2y$ that has two imaginary poles at $-1 + 2j$ and $-1 - 2j$. The poles form a conjugate pair. Bottom right: pole zero plot for the equation $Dy = 5x - 2Dx + D^2x$ that models a PID controller (2.2.7). The system has a single pole at the origin and two zeros at $1 + 2j$ and $1 - 2j$.

where r_ℓ are positive integers, A_ℓ are complex constants, K is a subset of the indices from $\{d, d+1, \dots, k\}$, and $p(s)$ is a polynomial of degree $m-k$. If $k > m$ then $p(s) = 0$. The integer r_ℓ is called the **multiplicity** of the pole β_ℓ. We see that the transfer function contains the summation of two parts: the polynomial $p(s)$, and a sum of terms of the form $\frac{A}{(s-\beta)^r}$. Let $p(s) = \gamma_0 + \gamma_1 s + \dots + \gamma_{m-k}s^{m-k}$. This polynomial is the transfer function of the nonregular system

$$F = \gamma_0 T_0 + \gamma_1 D + \gamma_2 D^2 + \dots + \gamma_{m-k} D^{m-k}.$$

This system is a linear combination of the identity system T_0 and differentiators of order at most $m-k$. From (4.2.3),

$$\mathcal{L}\left(\frac{A}{r!}t^{r-1}e^{\beta t}u(t)\right) = \frac{A}{(s-\beta)^r} \qquad \operatorname{Re} s > \operatorname{Re} \beta$$

and so the terms of the form $\frac{A}{(s-\beta)^r}$ correspond with the transfer function of a regular system with impulse response $\frac{A}{r!}t^{r-1}e^{\beta t}u(t)$. Other signals with Laplace transform $\frac{A}{(s-\beta)^r}$ are discarded because they do not correspond with the impulse response of a causal system. Thus, $\sum_{\ell \in K} \frac{A_\ell}{(s-\beta_\ell)^{r_\ell}}$ is the transfer function of the regular system G with impulse response

$$g(t) = u(t) \sum_{\ell \in K} \frac{A_\ell}{r_\ell!} t^{r_\ell - 1} e^{\beta_\ell t}. \tag{4.6.2}$$

The system H mapping x to y is the sum of the regular system G and nonregular system F, that is,

$$y = Hx = Fx + Gx.$$

Observe that H is regular only if the system $F = 0$, that is, only if F maps all input signals to the signal $x(t) = 0$ for all $t \in \mathbb{R}$. This occurs only when the polynomial $p(s) = 0$, that is, only when the number of poles exceeds the number of zeros. The system H will be stable if both F and G are stable. Because the differentiator D^ℓ is not stable (Exercise 1.17) the system F is stable if only if the order of the polynomial $p(s)$ is zero, that is, if $p(s) = \gamma_0$ is a constant (potentially $\gamma_0 = 0$). In this case $Fx = \gamma_0 T_0 x$ is the identity system multiplied by a constant. The polynomial $p(s)$ is a constant only when the order of the denominator polynomial is greater than or equal to the order of the numerator polynomial, that is, when the number of poles is greater than or equal to the number of zeros. The regular system G is stable if and only if its impulse response g is absolutely integrable. This occurs only when the terms $e^{\beta_\ell t}$ inside the sum (4.6.2) are decreasing as $t \to \infty$, that is, only if the real part of the poles $\operatorname{Re} \beta_\ell$ are negative. Thus, the system G is stable if and only if the real part of the poles are strictly negative.

The stability of the system H can be immediately determined from its pole-zero plot. The system is stable if and only if:

1. the number of poles is greater than or equal to the number of zeros (there are at least as many crosses on the pole-zero plot as circles),
2. No poles (crosses) line on the imaginary axis or in the right half of the complex plane.

The pole-zero plots in Figure 4.6 all represent stable systems with the exception of the plot on the bottom right (a PID controller). This system has two zeros and only one pole. The single pole is contained on the imaginary axis.

4.6.1 Two masses, a spring, and a damper

Consider the system involving two masses, a spring, and a damper in Figure 2.11. From (2.3.2), the equation relating the force applied to the first mass f and the position of the second mass p is

$$f = BDp + (M_1 + M_2)D^2 p + \frac{BM_2}{K} D^3 p + \frac{M_1 M_2}{K} D^4 p,$$

where B is the damping coefficient, K is the spring constant, and M_1 and M_2 are the masses. The transfer function of a system H that maps f to p is

$$\lambda H(s) = \frac{1}{s\left(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1 M_2}{K}s^3\right)}.$$

The system has no zeros and 4 poles. One of these poles always exists at the origin. The system is not stable because this pole is not strictly in the left half of the complex plane.

Consider the specific case when $B = K = M_1 = M_2 = 1$. Factorising the denominator polynomial gives

$$\lambda H(s) = \frac{1}{s(s - \beta_1)(s - \beta_2)(s - \beta_2^*)},$$

where

$$\beta_1 = \frac{1}{3}\left(\gamma - \frac{5}{\gamma} - 1\right) \approx -0.56984,$$

$$\beta_2 = \frac{1}{6}\left(\frac{5(1 + j\sqrt{3})}{\gamma} - (1 - j\sqrt{3})\gamma - \frac{1}{2}\right) \approx -0.21508 + 1.30714j,$$

and $\gamma = \left(\frac{3\sqrt{69} - 11}{2}\right)^{1/3}$. Applying partial fractions (Exercise 4.10) gives

$$\lambda(H) = \frac{1}{s(s - \beta_1)(s - \beta_2)(s - \beta_2^*)} = \frac{A_0}{s} + \frac{A_1}{s - \beta_1} + \frac{A_2}{s - \beta_2} + \frac{A_2^*}{s - \beta_2^*},$$

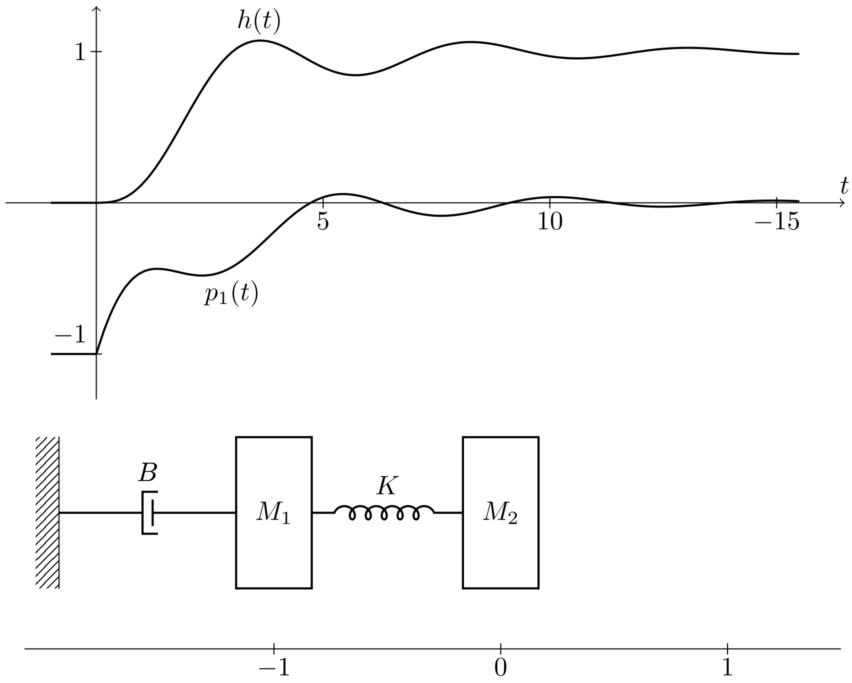

Figure 4.7: Impulse response of the system with two masses, a spring, and a damper, where $B = K = M_1 = M_2 = 1$.

where

$$A_0 = -\frac{1}{\beta_1|\beta_2|^2} = 1, \qquad A_1 = \frac{1}{\beta_1|\beta_1 - \beta_2|^2} \approx -0.956611,$$

$$A_2 = \frac{1}{\beta_2(\beta_2 - \beta_1)(\beta_2 - \beta_2^*)} \approx -0.0216944 + 0.212084j.$$

From (4.6.2), the impulse response of H is

$$h(t) = u(t)\big(A_0 + A_1 e^{\beta_1 t} + 2\,|A_2|\,e^{\mathrm{Re}\,\beta_2 t}\cos(\mathrm{Im}\,\beta_2 t + \angle A_2)\big).$$

This impulse response is plotted in Figure 4.7. Observe that h is not absolutely integrable and the system is not stable. The impulse response $h(t)$ does not converge to zero as $t \to \infty$ and correspondingly the mass M_2 does not come come to rest at position zero in Figure 4.7. In the figure it is assumed that the spring is at equilibrium when the two masses are $d = 1$ apart. From (2.3.1), the position of mass M_1 is given by the signal $p_1 = g - d$ where $g = h + M_2 D^2(h)$.

4.6.2 Direct current motors

Recall the direct current (DC) motor from Figure 2.13 described by the differential equation from (2.4.1),

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta,$$

where v is the input voltage signal and θ is a signal representing the angle of the motor. The constants R, B, K_τ, K_b, and J are related to components of the motor as described in Section 2.4. To simplify the differential equation put $a = \frac{RB}{K_\tau} + K_b$ and $b = \frac{RJ}{K_\tau}$ and the equation becomes

$$v = aD\theta + bD^2\theta.$$

The transfer function of a system H that maps input voltage v to motor angle θ is

$$\lambda H(s) = \frac{1}{s(a + bs)}.$$

This system has no zeros and two poles. One pole is at $-\frac{a}{b}$ and the other is at the origin. The system is not stable because the pole at the origin is not strictly in the left half of the complex plane.

Applying partial fractions we find that

$$\lambda H(s) = \frac{1}{as} - \frac{1}{a(s - \beta)}, \tag{4.6.3}$$

where $\beta = -\frac{a}{b}$. Using (4.2.3), the impulse response of H is

$$h(t) = \frac{1}{a} u(t)\big(1 - e^{\beta t}\big). \tag{4.6.4}$$

Other signals with Laplace transform (4.6.3) are discarded because they do not lead to a causal system. The step response Hu is obtained by applying the integrator system I_∞ to the impulse response, that is

$$Hu = I_\infty h = \frac{1}{a\beta} u(t)\big(\beta t + e^{\beta t} - 1\big).$$

The impulse response and step response are plotted in Figure 4.8 when $K_b = \frac{1}{8}$, $K_\tau = 8$ and $B = R = 1$ and $J = 2$ so that $a = \frac{1}{4}$, $b = \frac{1}{4}$ and $\beta = -1$.

Exercises

4.1. Sketch the signal

$$x(t) = e^{-2t} u(t) + e^t u(-t)$$

where $u(t)$ is the step function. Find the Laplace transform of $x(t)$ and the corresponding region of convergence. Sketch the region of convergence on the complex plane.

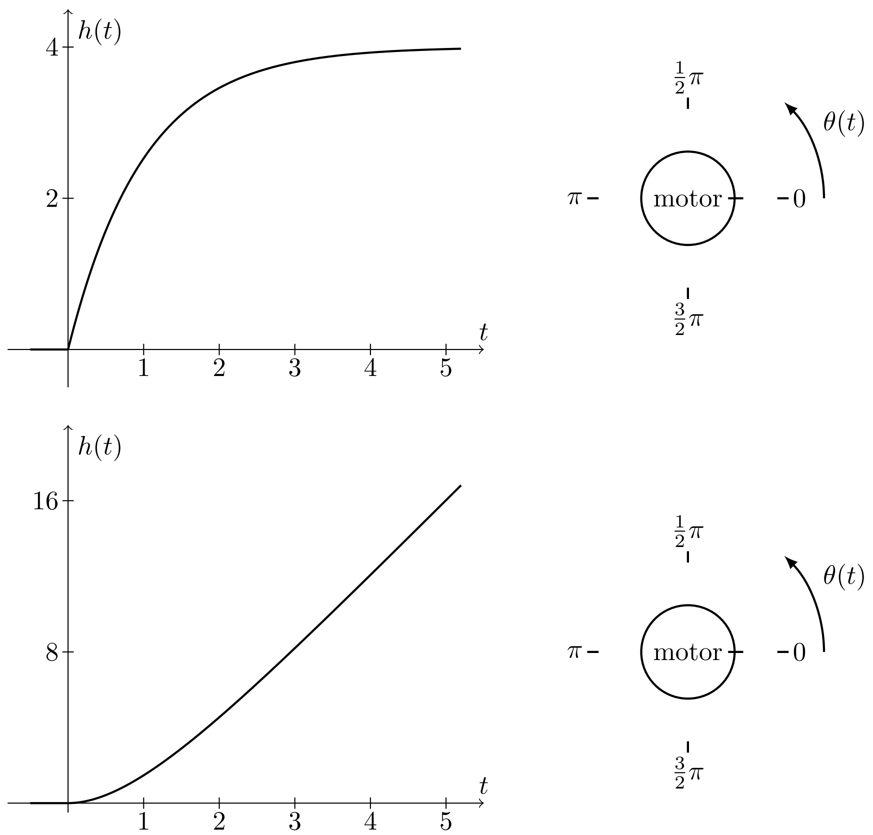

Figure 4.8: Impulse response (top) and step response (bottom) of a DC motor with constants $K_b = \frac{1}{4}$, $K_\tau = 8$ and $B = R = J = 1$.

4.2. Find the Laplace transform of the signal $t^n u(t)$ where $n \geq 0$ is an integer.

4.3. Let $n \geq 0$ be an integer. Show that the Laplace transform of the signal $(-t)^n u(-t)$ is the same as the Laplace transform of the signal $t^n u(t)$, but with a different region of convergence.

4.4. Show that equation (4.3.5) on page 57 holds when the system H the time shifter T_τ.

4.5. Show that equation (4.3.5) on page 57 holds when the system H is the differentiator under the added assumption that

$$\lim_{t\to\infty} x(t)e^{-st} = \lim_{t\to-\infty} x(t)e^{-st} = 0 \qquad \text{when } s \in \operatorname{roc}(x).$$

4.6. Let x be the signal with Laplace transform

$$\mathcal{L}(x, s) = \frac{1}{(s-1)^3} \qquad \operatorname{Re}(s) > 1.$$

Define the signal y by

$$y(t) = e^t x(2t+1).$$

Find the Laplace transform and region of convergence of y. Sketch the region of convergence of y.

4.7. What is the transfer function of the integrator system I_∞? What is the domain of this transfer function?

4.8. By partial fractions, or otherwise, assert that

$$\frac{as}{s+b} = a - \frac{ab}{s+b}$$

4.9. By partial fractions, or otherwise, assert that

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}$$

∗4.10. By partial fractions, or otherwise, assert that

$$\frac{1}{s(s-a)(s-b)(s-b^*)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_2^*}{s-b^*}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$ and $\operatorname{Im}(b) \neq 0$ and

$$A_0 = -\frac{1}{a|b|^2}, \qquad A_1 = \frac{1}{a|a-b|^2}, \qquad A_2 = \frac{1}{b(b-a)(b-b^*)}.$$

You might wish to check your solution using a symbolic programming language (for example Sage, Mathematica, or Maple).

4.11. Let y be a signal with Laplace transform taking the form

$$\mathcal{L}y(s) = \frac{2s+1}{s^2+s-2}$$

By partial fractions, or otherwise, find all possible signals y with this Laplace transform and the corresponding region of convergence.

4.12. Let x be a signal. Show that the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ satisfies equation (4.2.4) on page 55.

4.13. Consider the active electrical circuit from Figure 2.8 described by the differential equation from (2.2.3). Derive the transfer function of this system. Find an explicit system H that maps the input voltage x to the output voltage y. State whether this system is stable and/or regular.

∗4.14. Given the mass spring damper system described by (4.5.1), find the position signal p given that the force signal

$$f(t) = \Pi\big(t - \tfrac{1}{2}\big) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the rectangular function time shifted by $\frac{1}{2}$. Consider three cases:

(a) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \frac{\pi}{3}$,
(b) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \pi$,
(c) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = 2\pi$,

Plot the solution in each case, and comment on whether the system is underdamped, overdamped, or critically damped.

4.15. Plot the signal $x(t) = \sin(te^t)u(t)$ and find and plot its derivative Dx. Show that the region of convergence of x contains those complex numbers s with $\operatorname{Re} s > 0$ and that the region of convergence of Dx contains those with $\operatorname{Re} s > 1$.

4.16. Show that the limit as $|s| \to 0$ of

$$\frac{e^{s/2} - e^{-s/2}}{s}$$

is equal to 1.

∗4.17. Consider the mechanical system in Figure 2.15 from Exercise 2.2. After solving Exercise 2.2, find the transfer function of a linear shift-invariant H system mapping f to p. Now suppose that $M_1 = K_1 = K_2 = B = 1$ and $M_2 = 2$. Find the poles and zeros of H and draw a pole zero plot. Determine whether H is stable and/or regular. Find and plot the impulse response and the step response of H if they exist.

4.18. Consider the electromechanical system in Figure 2.16 from Exercise 2.3. After solving Exercise 2.3, find the transfer function of a linear shift-invariant system that maps the input voltage v to the motor angle θ. Under the assumption that the motor coefficients satisfy $L = 0$ and $K_b = K_\tau = B = R = J = 1$ draw a pole zero plot and determine whether this system is stable and/or regular. Find and plot the impulse response and step response if they exist.

$**$4.19. Let x be a signal. Show that the complex exponential signal $e^{st} \in \operatorname{dom} x$ if and only if the signal $x(t)e^{-st}$ is absolutely integrable.

$**$4.20. Show that the complex exponential signal $e^{st} \in \operatorname{dom} f\ g$ if and only if $s \in \operatorname{roc} f \cap \operatorname{roc} g$, that is, $\operatorname{cep} \operatorname{dom} f\ g = \operatorname{roc} f \cap \operatorname{roc} g$.

Chapter 5

The Fourier transform

Let x be an absolutely integrable signal. We denote by $\mathcal{F}x$ the complex valued function satisfying

$$\mathcal{F}x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{5.0.1}$$

called the **Fourier transform** of x. The Fourier transform is a complex valued function of the real number f, that is, $\mathcal{F}x \in \mathbb{R} \to \mathbb{C}$, or in other words, $\mathcal{F}x$ is a **signal**. For example, the rectangular pulse $\Pi(t)$ from (1.1.2) is absolutely integrable and has Fourier transform

$$\begin{aligned}
\mathcal{F}\Pi(f) &= \int_{-\infty}^{\infty} \Pi(t)e^{-j2\pi ft}dt \\
&= \int_{-1/2}^{1/2} e^{-j2\pi ft}dt \\
&= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} = \operatorname{sinc}(f).
\end{aligned} \tag{5.0.2}$$

The sinc function is plotted in Figure 5.1.

The Fourier transform is closely related to the Laplace transform because

$$\mathcal{F}x(f) = \mathcal{L}x(j2\pi f)$$

for those signals x with region of convergence containing the imaginary axis, that is, for absolutely integrable x. The Fourier transform inherits the properties of the Laplace transform that were described in Section 4.3. For example, if H is a stable regular system with absolutely integrable impulse response h (Exercise 3.9), then the spectrum of H satisfies

$$\Lambda H(f) = \lambda H(j2\pi f) = \mathcal{L}h(j2\pi f) = \mathcal{F}h(f),$$

that is, the spectrum of a stable regular system is the Fourier transform of its impulse response. Like the Laplace transform, the Fourier transform obeys the **convolution theorem** (4.3.4), that is,

$$\mathcal{F}(x * y) = \mathcal{F}x\mathcal{F}y \tag{5.0.3}$$

when the signals x and y are absolutely integrable. In words: the Fourier transform of a convolution of signals is the multiplication of the Fourier transforms of those signals. The convolution of two absolutely integrable signals is always absolutely integrable and so there is no need to include the assumption that $x * y$ is absolutely integrable (Exercises 3.9).

It follows from (4.3.5) that if H is a stable regular system with impulse response h and spectrum $\Lambda H = \mathcal{F}h$ and if $x \in \operatorname{dom} h$ is a signal with Fourier transform $\mathcal{F}x$, then the signal Hx has Fourier transform

$$\mathcal{F}Hx = \Lambda H \, \mathcal{F}x, \tag{5.0.4}$$

that is, the Fourier transform of Hx is the multiplication of the transfer function of the system H and the Fourier transform of the input signal x. This property also holds for the shifter T_τ (Exercise 4.4) and it holds for the differentiator D under the added assumption that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to-\infty} x(t) = 0$ (Exercise 4.5). From (4.3.1) and (4.3.2) the spectrum of T_τ and the kth differentiator D^k satisfy

$$\Lambda T_\tau = e^{-j2\pi f\tau}, \qquad \Lambda D^k = (j2\pi f)^k$$

from which we obtain the **time shift property**,

$$\mathcal{F}T_\tau x = \Lambda T_\tau \mathcal{F}x = e^{-j2\pi f\tau}\mathcal{F}x,$$

and the **differentiation property**,

$$\mathcal{F}D^k x = \Lambda D^k \mathcal{F}x = (j2\pi f)^k \mathcal{F}x.$$

These results motivate assigning the following Fourier transforms to the delta "function" δ, its shift $T_\tau\delta = \delta(t-\tau)$, and its derivatives

$$\mathcal{F}\delta = 1, \qquad \mathcal{F}\big(\delta(t-\tau)\big) = e^{-j2\pi\tau f}, \qquad \mathcal{F}\delta^k = (j2\pi f)^k. \tag{5.0.5}$$

These conventions are common in the literature [Oppenheim et al., 1996].

Similarly to the Laplace transform (4.2.2), the Fourier transform obeys a **frequency shift rule** that relates the transform of a signal $x(t)$ to that of the signal $e^{2\pi j\gamma f t}x(t)$ where $\gamma \in \mathbb{R}$. From (4.2.2), the frequency shift rule asserts that

$$\mathcal{F}\big(e^{2\pi j\gamma t}x(t)\big)(f) = \mathcal{F}x(f-\gamma), \tag{5.0.6}$$

that is, the Fourier transform of the signal $e^{2\pi j\gamma f t}x(t)$ is given by shifting that of x by γ. The property can be expressed using the shifter system T_γ

by $\mathcal{F}\big(e^{2\pi j\gamma t}x(t)\big) = T_\gamma \mathcal{F}x$. The signal $e^{2\pi j\gamma f t}x(t)$ is often referred to as a frequency shifted version of x.

Because $\cos(2\pi\gamma t) = \frac{1}{2}e^{2\pi j\gamma t} + \frac{1}{2}e^{-2\pi j\gamma t}$ it follows from the frequency shift rule that

$$\mathcal{F}\big(\cos(2\pi\gamma t)x(t)\big)(f) = \frac{1}{2}\mathcal{F}x(f-\gamma) + \frac{1}{2}\mathcal{F}x(f+\gamma). \tag{5.0.7}$$

and, similarly, since $\sin(2\pi\gamma t) = \frac{1}{2j}e^{2\pi j\gamma t} - \frac{1}{2j}e^{-2\pi j\gamma t}$ we have

$$\mathcal{F}\big(\sin(2\pi\gamma t)x(t)\big)(f) = \frac{1}{2j}\mathcal{F}x(f-\gamma) - \frac{1}{2j}\mathcal{F}x(f+\gamma).$$

These results are sometimes called the **modulation properties** of the Fourier transform [Papoulis, 1977, page 61]. These properties are of particular importance in communications engineering [Proakis, 2007]. Combing the frequency shift rule with the convention $\mathcal{F}\delta = 1$ motivates assigning the following Fourier transforms to the complex exponential signal $e^{2\pi j\gamma t}$ and the cosine and sine signals,

$$\mathcal{F}(e^{2\pi j\gamma t}) = \delta(f-\gamma).$$

$$\mathcal{F}\big(\cos(2\pi\gamma t)\big) = \tfrac{1}{2}\delta(f-\gamma) + \tfrac{1}{2}\delta(f+\gamma),$$
$$\mathcal{F}\big(\sin(2\pi\gamma t)\big) = \tfrac{1}{2j}\delta(f-\gamma) - \tfrac{1}{2j}\delta(f+\gamma).$$

These conventions are common in the literature [Oppenheim et al., 1996; Proakis, 2007], but must be treated with caution. There is no guarantee that mechanical mathematical manipulations involving these conventions will lead to sensible results.

Like the Laplace transform (4.2.4), the Fourier transform obeys a **time-scaling property**. If x is an absolutely integrable signal then the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ has Fourier transform

$$\mathcal{F}\big(x(\alpha t)\big)(f) = \frac{1}{|\alpha|}\mathcal{F}x(f/\alpha). \tag{5.0.8}$$

5.1 The inverse transform and the Plancherel theorem

Given a signal x we will often denote its Fourier transform by $\hat{x} = \mathcal{F}x$. Observe that $\hat{x}$, like x, is a function that maps a real number to a complex number, that is, $\hat{x}$ is a **signal** with independent variable representing frequency. It is usual to call $\hat{x}$ the **frequency-domain** representation of the signal and x the **time-domain** representation although the signal x need not be a function of "time".

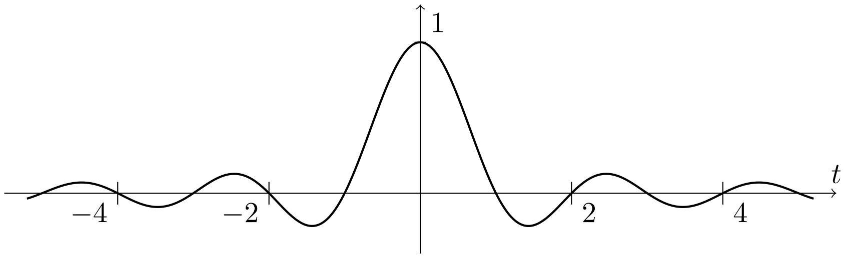

Figure 5.1: The **sinc function** $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

If $\hat{x}$ is absolutely integrable, then x can be recovered using the **inverse Fourier transform**

$$x(t) = \mathcal{F}^{-1}\hat{x}(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{j2\pi f t} df. \tag{5.1.1}$$

For example, suppose that $\hat{x} = \mathcal{F}x = \Pi$ is the rectangular pulse. By working analogous to that from (5.0.2),

$$x(t) = \int_{-\infty}^{\infty} \Pi(f) e^{j2\pi f t} df = \operatorname{sinc}(-t) = \operatorname{sinc}(t).$$

We are lead to the conclusion that the Fourier transform of sinc is the rectangular pulse Π.

The rectangular pulse Π is finite and absolutely integrable. The sinc function is not absolutely integrable (Exercise 5.3). Because of this the integral equation that we have used to define the Fourier transform (5.0.1) cannot be directly applied to the sinc function. Although sinc is not absolutely integrable, it is square integrable (Exercise 5.3). It happens that all square integrable signals can be assigned a Fourier transform by interpreting the integral in (5.0.1) as what is called its **Cauchy principal value**. That is, for x a square integrable signal, we assign the Fourier transform

$$\hat{x}(f) = \mathcal{F}x(f) = \lim_{T\to\infty} \int_{-T}^{T} x(t) e^{-j2\pi f t} dt. \tag{5.1.2}$$

This Fourier transform $\hat{x}$ is itself a square integrable signal and the orignal time domain signal x can be recovered almost everywhere by taking the Cauchy principal value of the integral for the inverse Fourier transform (5.1.1), that is,

$$x(t) = \mathcal{F}^{-1}\hat{x}(t) = \lim_{T\to\infty} \int_{-T}^{T} \hat{x}(f) e^{j2\pi f t} df \qquad \text{a.e.}$$

Infact, the energy of x and its Fourier transform $\hat{x}$ are the same, that is,

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, dt = \|\hat{x}\|_2^2. \tag{5.1.3}$$

These results are known as the **Plancherel theorem** [Rudin, 1986, Th. 9.13]. The equality of energies in (5.1.3) is often called **Parseval's identity**. For our purposes it will suffice to remember only that the Fourier transform of the sinc function is the rectangular pulse Π. In this text, the sinc function is by far the most regularly occuring example of a signal that is square integrable, but not absolutely integrable. If x and y are square integrable signals with Fourier transforms $\hat{x}$ and $\hat{y}$, then a consequence of (5.1.3) is

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} \hat{x}(f)\hat{y}^*(f)dt \qquad \text{(Exercise 5.15)} \qquad (5.1.4)$$

where the superscript * denotes the complex cojugate. This result often also goes by the name of Parseval's identity.

Let x be a signal with Fourier transform

$$\mathcal{F}x(f) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau.$$

Evaluating the Fourier transform at $-t$ we find that

$$\mathcal{F}x(-t) = \int_{-\infty}^{\infty} x(\tau)e^{j2\pi t\tau}d\tau = \mathcal{F}^{-1}x(t). \qquad (5.1.5)$$

This is the called the **duality** property of the Fourier transform. In words, if $\hat{x}$ is the Fourier transform of x, then x is the Fourier transform of $\hat{x}$ reflected in time. Another way to express duality is

$$\mathcal{F}\hat{x}(t) = \mathcal{F}\mathcal{F}x(t) = \mathcal{F}^2x(t) = x(-t) \qquad \text{a.e.,}$$

that is, twice application of the Fourier transform to a signal x results in x reflected in time. Pointwise equality holds in the case that $\hat{x}$ is absolutely integrable.

Suppose that x and y are square integrable with absolutely integrable Fourier transforms $\hat{x}$ and $\hat{y}$. Because $\mathcal{F}\hat{x}(t) = x(-t)$ and $\mathcal{F}\hat{y}(t) = y(-t)$, the convolution property asserts that

$$\mathcal{F}(\hat{x} * \hat{y})(-t) = x(t)y(t).$$

The product xy is an absolutely integrable signal by **Holder's inequality** [Rudin, 1986, Theorem 3.5]. Applying the Fourier transform to both sides and using the duality property we find that

$$\mathcal{F}(xy) = \hat{x} * \hat{y} \qquad (5.1.6)$$

This is called the **multiplication property** of the Fourier transform. In words, the Fourier transform of a multiplication of signals is the convolution of the Fourier transforms of those signals. The multiplication property can be shown to hold under only the assumption that x and y are square integrable provided that equality in (5.1.6) is replaced by equality almost everywhere (Exercise 5.16).

5.2 Analogue filters

For many engineering purposes it is desirable to construct systems that will **pass** (have little affect on) a complex exponential signal $e^{j2\pi ft}$ for certain frequencies f, but will **reject** (significantly attenuate) these signals for other frequencies. Such systems are called **frequency dependent filters**. Those frequencies that the filter intends to pass unaffected are said to be in the **pass band** and those frequencies that the filter intends to reject are said to be in the **stop band**.

An **ideal lowpass filter** with **cuttoff frequency** c is the system L_c with spectrum

$$\Lambda L_c = \begin{cases} 1 & |f| < c \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{f}{2c}\right).$$

Applying the inverse Fourier transform to $\Pi\left(\frac{f}{2c}\right)$ gives

$$\int_{-\infty}^{\infty} \Pi\left(\tfrac{f}{2c}\right) e^{j2\pi tf} df = \int_{-c}^{c} e^{j2\pi tf} df = \frac{\sin(2c\pi t)}{\pi t} = 2c\,\text{sinc}(2ct).$$

We are lead to the conclude that the ideal lowpass filter L_c is a regular linear time invariant system with impulse response $2c\,\text{sinc}(2ct)$. This conclusion should be taken with scepticism because the signal $2c\sin(2ct)$ is not abolutely integrable and so its transfer function with not contain the imaginary axis in its domain. Infact, $\text{roc}\,\text{sinc} = \emptyset$. An exception has been made to allow square integrable signals to have a Fourier transform and a similar exception must be made to allow regular systems with square integrable impulse response to have a spectrum. Ideal filters are not practically implementable and so this caveat with not affect the results that follow. The concept of an ideal filter is nonethless informative from an intuitive viewpoint.

An **ideal highpass filter** with cuttoff frequency c is given by the linear combination $T_0 - L_c$ where T_0 is the identity system. The spectrum is

$$\Lambda(T_0 - L_c) = \Lambda T_0 - \Lambda L_c = 1 - \Pi\left(\frac{f}{2c}\right) = \begin{cases} 0 & |f| < c \\ 1 & \text{otherwise.} \end{cases}$$

This ideal highpass filter is not regular because the system T_0 is not regular. The system does not have an impulse response. Nevertheless, it is common to represent one by $\delta(t) - 2c\,\text{sinc}(2ct)$ using the delta function as described in Section 3.1.

An **ideal bandpass filter** with upper cuttoff frequency u and lower cuttoff frequency ℓ is given by the linear combination $L_u - L_\ell$. The spectrum is

$$\Lambda(L_u - L_\ell) = \Pi\left(\frac{f}{2u}\right) - \Pi\left(\frac{f}{2\ell}\right) = \begin{cases} 1 & -u < f \leq -\ell \\ 1 & \ell \leq f < u \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that the ideal bandpass filter has impulse response $2u\,\mathrm{sinc}(2ut) - 2\ell\,\mathrm{sinc}(2\ell t)$. The spectrum and impulse response of the ideal lowpass, highpass, and bandpass filters are plotted in Figure 5.2.

Ideal filters are not realisable in practice. We now describe a popular practical low-pass filter discovered by Butterworth [1930]. A **normalised low pass Butterworth filter** of order m, denoted by B_m, has transfer function

$$\lambda B_m(s) = \frac{1}{\prod_{i=1}^{m}(\frac{s}{2\pi} - \beta_i)} = \frac{(2\pi)^m}{\prod_{i=1}^{m}(s - 2\pi\beta_i)},$$

where $\beta_1, \dots, \beta_m$ are the roots of the polynomial $s^{2m} + (-1)^m$ that lie strictly in the left half of the complex plane (have negative real part). Specifically, these roots are

$$\beta_k = \begin{cases} \exp\left(j\frac{\pi}{2}(1 + \frac{2k-1}{m})\right), & k = 1, \dots, m \\ \exp\left(j\frac{\pi}{2}(1 - \frac{2k-1}{m})\right), & k = m+1, \dots, 2m. \end{cases}$$

The roots are plotted in Figure 5.3. The spectrum of B_m is

$$\Lambda B_m(f) = \frac{1}{\prod_{i=1}^{m}(jf - \beta_i)}$$

and the magnitude spectrum of B_m can be shown to satisfy

$$|\Lambda B_m(f)| = \sqrt{\frac{1}{f^{2m}+1}}. \qquad \text{(Exercise 5.4)}$$

The magnitude and phase spectrum of the filters B_1, B_2, B_3, and B_4 are plotted in Figure 5.4.

The **cuttoff frequency** of the lowpass filter B_m is defined as the positive real number c such that $|\Lambda B_m(f)|^2 < \frac{1}{2}$ for all $f > c$. The normalised Butterworth filters have cuttoff frequency $c = 1$Hz. A lowpass Butterworth filter of order m and cuttoff frequency c, denoted B_m^c, has transfer function

$$\lambda B_m^c(s) = \lambda B_m\left(\tfrac{s}{c}\right) = \frac{1}{\prod_{i=1}^{m}(\frac{s}{2\pi c} - \beta_i)}.$$

The magnitude spectrum satisfies

$$|\Lambda B_m^c(f)|^2 = |\Lambda B_m\left(\tfrac{f}{c}\right)|^2 = \frac{1}{\left(\frac{f}{c}\right)^{2m} + 1} = \frac{c^{2m}}{f^{2m} + c^{2m}}. \qquad (5.2.1)$$

A first order Butterworth filter B_1^c has spectrum

$$\Lambda B_1^c(f) = \frac{1}{j\frac{f}{c} + 1} = \frac{c}{jf + c}.$$

Putting $\frac{1}{c} = 2\pi RC$ we find that this is the same as the spectrum of the RC electrical circuit (Figure 2.1) or the active RC circuit after negation (3.5.2).

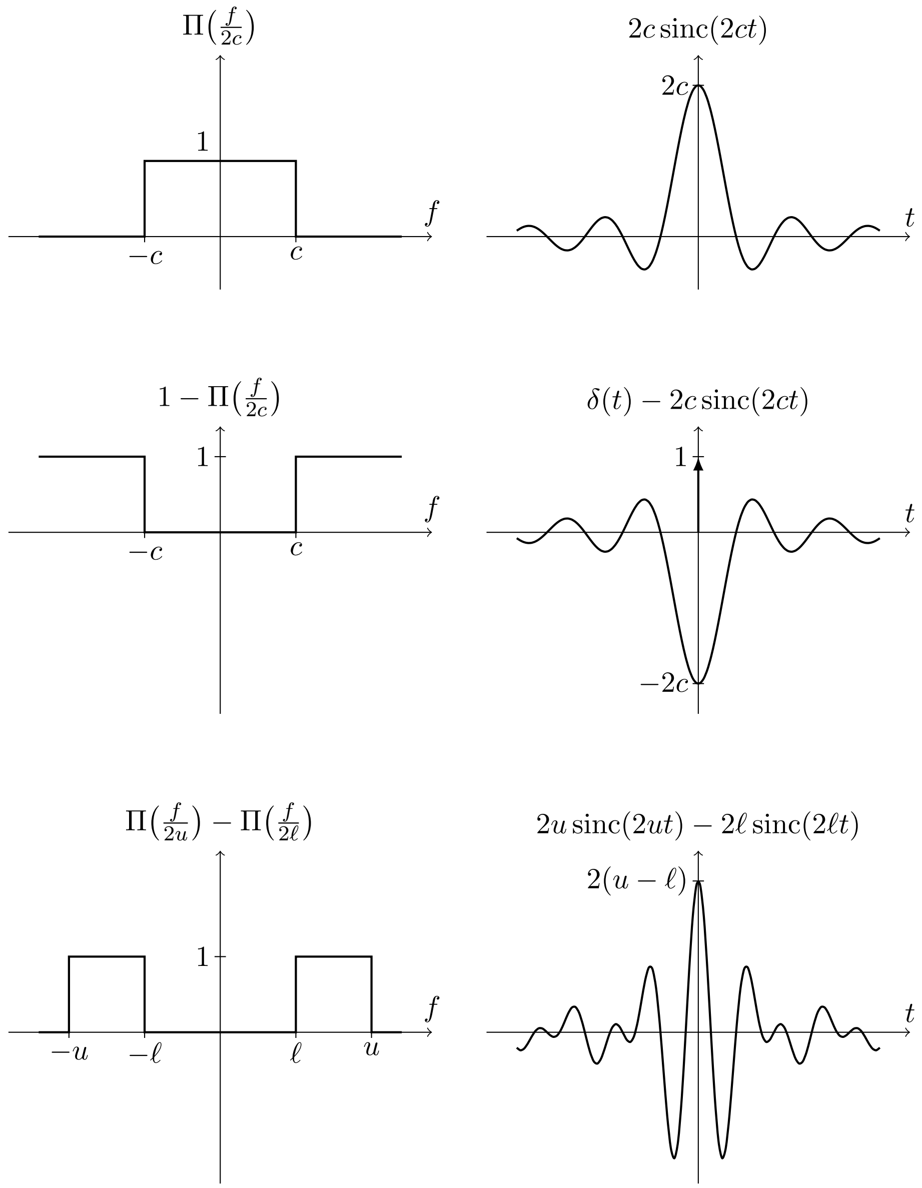

Figure 5.2: Spectrum and impulse response of the ideal lowpass filter L_c (top), the ideal highpass filter $T_0 - L_c$ (middle), and the ideal bandpass filter $L_u - L_\ell$ (bottom). The ideal highpass filter is not regular and does not have an impulse response. We plot the 'pretend' impulse response using the delta function described in Section 3.1.

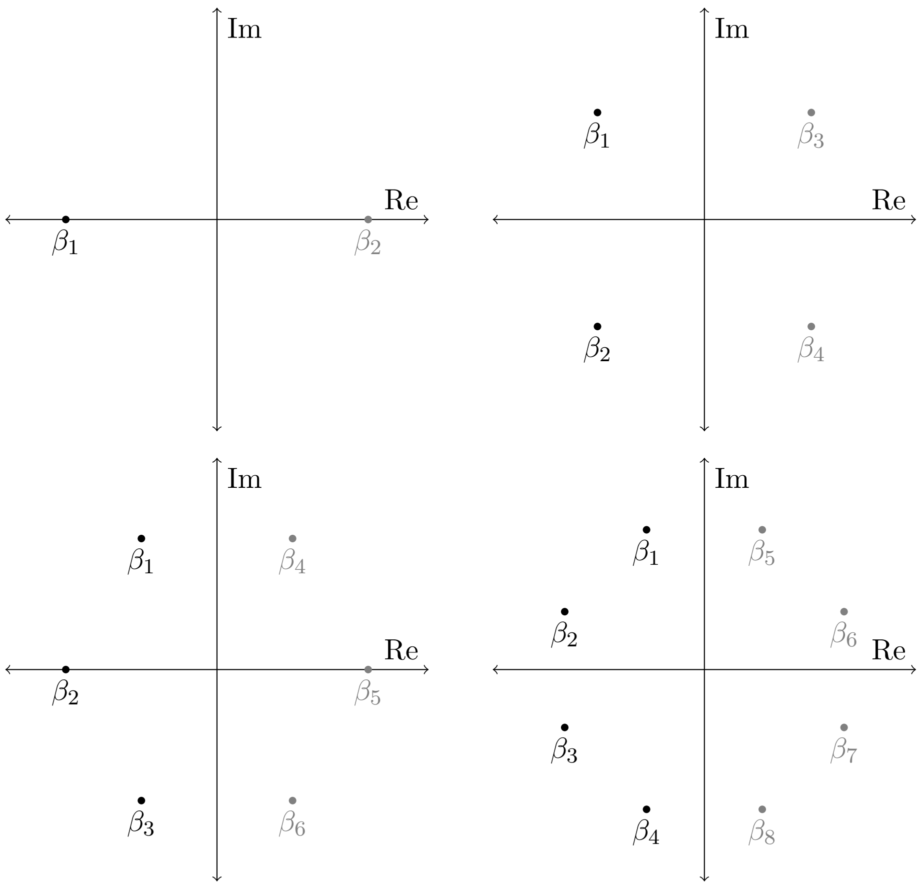

Figure 5.3: Roots of the polynomial $s^{2m} + (-1)^m$ for $m = 1$ (top left), $m = 2$ (top right), $m = 3$ (bottom left), and $m = 4$ (bottom right). All the roots lie on the complex unit circle and have magnitude one. The poles of the normalised Butterworth filter B_m are those roots from the left half of the complex plane (unshaded).

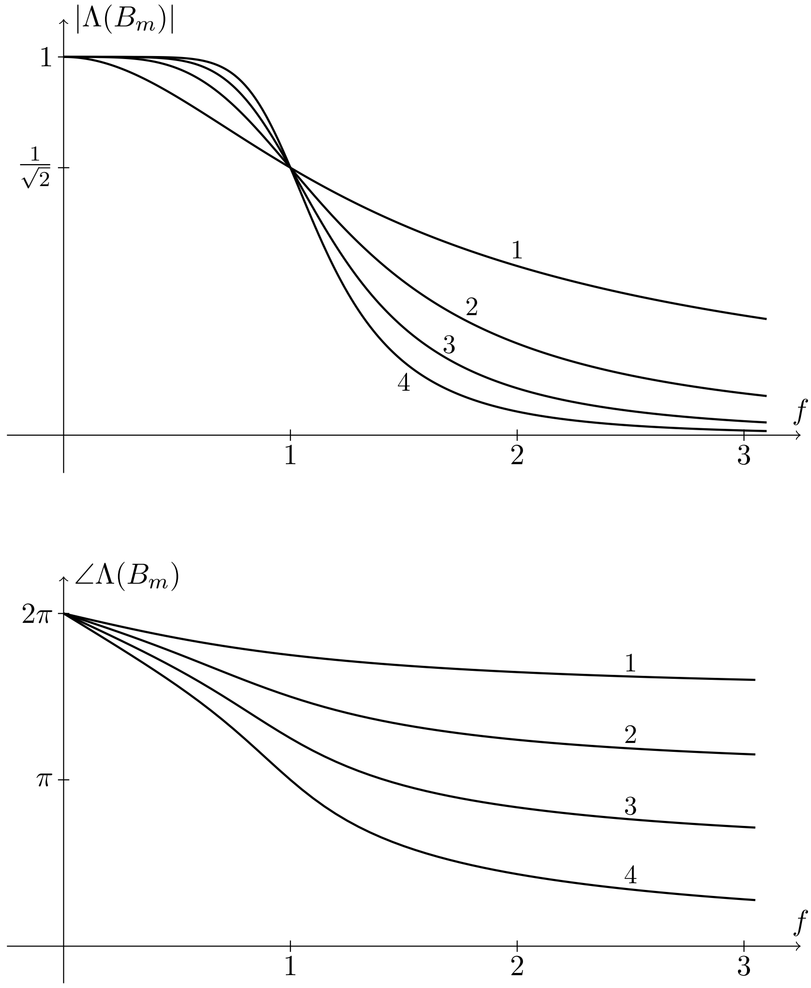

Figure 5.4: Magnitude spectrum (top) and phase spectrum (bottom) of normalised Butterworth filters B_1, B_2, B_3 and B_4.

Thus, the RC electrical circuit is a first order Butterworth filter with cuttoff frequency $c = \frac{1}{2\pi RC}$. In Test 4 we constructed the active RC circuit with $R \approx 27\text{k}\Omega$ and $C \approx 10\text{nF}$ and measured its magnitude spectrum. The cuttoff frequency was $c = \frac{5\times 10^4}{27\pi} \approx 589\text{Hz}$.

A second order electrical Butterworth filter can be constructed using the Sallen-Key circuit described in Section 2.2 and Figure 2.10. The input voltage x and output voltage y of the Sallen-Key satisfy the differential equation (2.2.9)

$$x = y + C_2(R_1 + R_2)Dy + R_1R_2C_1C_2D^2y.$$

The transfer function corresponding with this equation is

$$\frac{1}{1 + C_2(R_1 + R_2)s + R_1R_2C_1C_2s^2}.$$

The second order Butterworth filter B_2^c has transfer function

$$\Lambda B_2^c(s) = \frac{1}{(\frac{1}{2\pi c}s - \beta_1)(\frac{1}{2\pi c}s - \beta_2)},$$

where $\beta_1 = \beta_2^* = e^{j3\pi/4}$. Expanding the quadratic on the denominator gives

$$\Lambda B_2^c(s) = \frac{1}{1 + \frac{1}{\sqrt{2}\pi c}s + \frac{1}{4\pi^2c^2}s^2}.$$

Choosing the resistors and capacitors of the Sallen-Key to satisfy

$$C_2(R_1 + R_2) = \frac{1}{\sqrt{2}\pi c}, \qquad R_1R_2C_1C_2 = \frac{1}{4\pi^2c^2}$$

leads to a second order Butterworth filter. A convenient solution is to put $C_1 = 2C_2$ and $R_1 = R_2$. This gives a second order Butterworth filter with cuttoff

$$c = \frac{1}{\sqrt{2}\pi C_2(R_1 + R_2)} = \frac{1}{2\sqrt{2}\pi C_2R_2}.$$

In Test 6 we construct a second order Butterworth filter using a Sallen-Key and measure its spectrum.

Butterworth filters of orders larger than $m = 2$ can be constructed by concatenating Sallen-Key circuits and RC circuits. If m is even then $m/2$ Sallen-Key circuits are required. Each Sallen-Key is used to construct a conjugate pair of poles, that is, the kth Sallen-Key would have poles $2\pi c\beta_k$ and $2\pi c\beta_k^* = 2\pi c\beta_{m-k+1}$. If m is odd then $(m-1)/2$ Sallen-Key circuits and a single RC circuit (or active RC circuit) can be used. The RC circuit is designed to have the real valued pole $\beta_{(m+1)/2} = 2\pi c$.

Test 6 (**Butterworth filter**)
We construct a second order Butterworth filter using the Sallen-Key circuit from Figure 2.10 with capacitors $C_2 \approx 100\text{nF}$, $C_1 \approx 2C_2 \approx 200\text{nF}$ and resistors $R_1 \approx R_2 \approx 330\Omega$. The cuttoff frequency is

$$c = \frac{1}{2\sqrt{2}\pi C_2 R_2} \approx 3410\text{Hz}.$$

Sinusoids of the form

$$\sin(2\pi f_k t), \qquad f_k = \left\lceil 110 \times 2^{k/2} \right\rfloor, \quad k = 1, 2, \dots, 13$$

are input to the filter using a computer soundcard and the magnitude and phase spectrum are measured using the procedure described in Test 4. Figure 5.5 shows the measurements (dots) plotted alongside the hypothesised magnitude spectrum

$$|\Lambda B_2^c(f)| = \sqrt{\frac{1}{(f/c)^4 + 1}}$$

and the hypothesised phase spectrum $\angle \Lambda B_2^c(f)$.

5.3 Complex sequences

Let x be a signal with Fourier transform $\hat{x} = \mathcal{F}x$. The signal x is said to be **bandlimited** if there exists a positive real number b such that

$$\hat{x}(f) = \mathcal{F}x(f) = 0 \qquad \text{when } |f| > b.$$

The value b is called the **bandwidth** of the signal x. For example, the sinc function is bandlimited with bandwidth $\frac{1}{2}$ because its Fourier transform $\mathcal{F}\,\text{sinc}(f) = \Pi(f) = 0$ for all $|f| > \frac{1}{2}$. Bandlimited signals have a number of properties that make them suitable for representation and manipulation by a computer. They are of particular importance for this reason. Before we can study bandlimited signals we first require some properties of real and complex valued **sequences**.

A **sequence** is a function with domain given by the integers $\mathbb{Z}$. The value of a sequence x at integer n can be denoted by $x(n)$ but it is conventional to write x_n. We are primarily interested in sequences that take complex values, that is, functions from the set $\mathbb{Z} \to \mathbb{C}$. For example,

$$\sin(\tfrac{\pi}{4}n), \qquad n^3, \qquad e^{-|n|/2}$$

each denote a real (and so also complex) valued sequence. In what follows the term **sequence** will always mean a complex valued sequence unless otherwise stated. Complex valued sequences are commonly called **discrete-time**

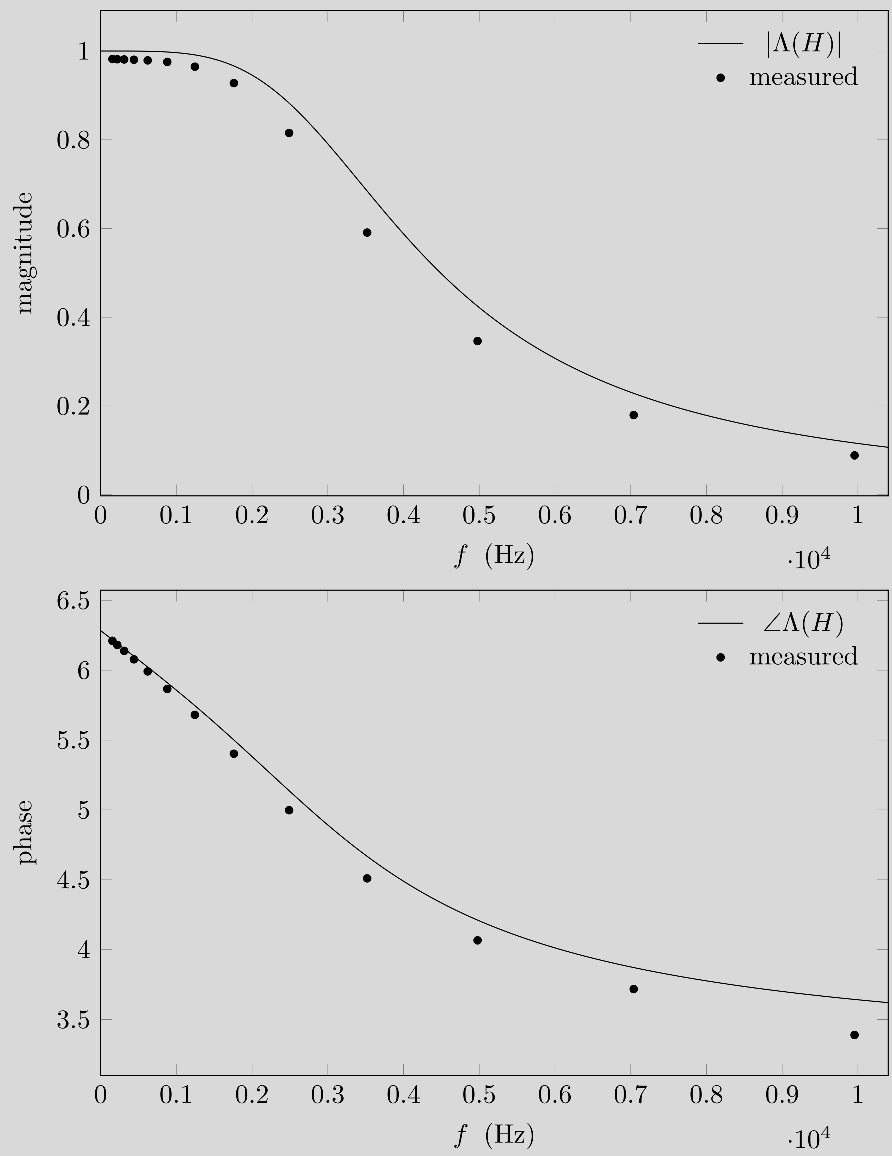

Figure 5.5: Hypothesised magnitude spectrum $|\Lambda(B_m^c)|$ (top) and phase spectrum $\angle\Lambda(B_m^c)$ (bottom) of the second order Butterworth filter and the measured magnitude and phase spectrum of the filter implemented with a Sallen-Key active electrical circuit (dots).

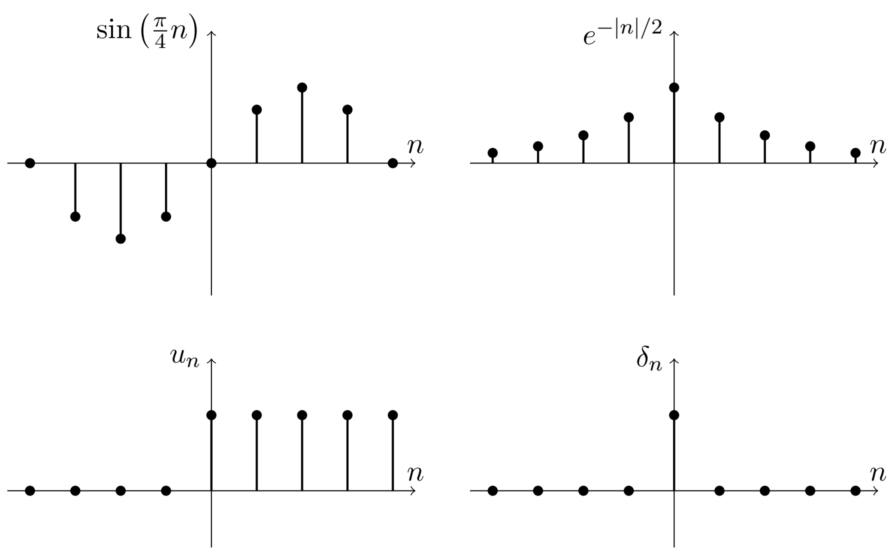

Figure 5.6: Real valued sequences. The bottom plots show that step sequence u and the delta sequence δ.

signals and the nth element in the sequence is denoted by $x[n]$ using squared brackets [Oppenheim et al., 1996]. Here, we use the subscript notation x_n. This notation is also common [Vetterli et al., 2014; Rudin, 1986]. Sequences are plotted using vertical lines with dotted ends as in Figure 5.6 and have a number of properties analogous to the properties of signals (Section 1.1).

A sequence x is bounded if there exists a real number M such that

$$|x_n| < M \qquad \text{for all } n \in \mathbb{Z}.$$

Both $\sin(\frac{\pi}{4}n)$ and $e^{-|n|/2}$ are examples of bounded sequences, but n^3 is not bounded because its magnitude grows indefinitely as n moves away from the origin. A sequence x is periodic if there exists a positive integer T such that

$$x_n = x_{n+kT} \qquad \text{for all integers } k \text{ and } n.$$

The smallest such T is called the period. The sequence $\sin(\frac{\pi}{4}n)$ is periodic with period $T = 8$. Neither n^3 or $e^{-n^2/4}$ are periodic. A sequence x is even (or symmetric) if $x_n = x_{-n}$ for all $n \in \mathbb{Z}$ and odd (or antisymmetric) if $x_n = -x_{-n}$ for all $n \in \mathbb{Z}$. Both $\sin(\frac{\pi}{4}n)$ and n^3 are odd and $e^{-|n|/2}$ is even.

A sequence x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x_n = 0$ for all $n < T$. Correspondingly x is **left sided** if $x_n = 0$ for all $n > T$. For example, the **step sequence** u with nth element

$$u_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.3.1}$$

is right sided (Figure 1.2). The reflected sequence u_{-n} is left sided. A sequence is said to be of **finite support** or just **finite** if it is both left and right sided. For example the the **delta sequence** δ with nth element

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise}, \end{cases} \tag{5.3.2}$$

has finite support. The delta sequence is analogous to the delta "function" introduced in Section 3.1. The delta "function" is not actually function, only a notational device. Contrastingly, the delta sequence is a well defined sequence.

A sequence x is **absolutely summable** if

$$\|x\|_1 = \sum_{n \in \mathbb{Z}} |x_n| < \infty,$$

that is, if the sum of absolute values of the elements in the sequence converges to a finite number. The real number $\|x\|_1$ is commonly called the ℓ^1-norm of x. The sequences $\sin(\frac{\pi}{4}n)$ and n^3 are not absolutely summable, but $e^{-|n|/2}$ is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}| = \sum_{n \in \mathbb{Z}} e^{-|n|/2} = 1 + \frac{2}{\sqrt{e} - 1}. \qquad \text{(Exercise 5.9)}$$

It is common to denote the set of absolutely summable sequences by ℓ^1 or $\ell^1(\mathbb{Z})$. So, $e^{-|n|/2} \in \ell^1$ and $\sin(\frac{\pi}{4}n) \notin \ell^1$.

A sequence x is **square summable** if

$$\|x\|_2^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty,$$

that is, if the sum of squared magnitudes of the elements converges to a finite number. The real number $\|x\|_2$ is commonly called the ℓ^2-norm and its square $\|x\|_2^2$ the **energy** of x. The sequences $\sin(\frac{\pi}{4}n)$ and n^3 are not square summable, but $e^{-|n|/2}$ is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}|^2 = \sum_{n \in \mathbb{Z}} e^{-|n|} = 1 + \frac{2}{e - 1}. \qquad \text{(Exercise 5.9)}$$

It is common to denote the set of square summable sequences by ℓ^2. So, $e^{-|n|/2} \in \ell^2$ and $\sin(\frac{\pi}{4}n) \notin \ell^2$. If a sequence is absolutely summable then it is also square summable (Exercise 5.10). The corresponding property is not true of signals, that is, absolutely integrable signals are not necessarily square integrable (Exercise 1.5).

5.4 Bandlimited signals

Let b be a positive real number and let x be a signal with Fourier transform $\hat{x} = \mathcal{F}x$. The signal x is said to be **bandlimited** with **bandwidth** b if

$$\hat{x}(f) = \mathcal{F}x(f) = 0 \qquad \text{for all } |f| > b.$$

For example, the sinc function $\text{sinc}(t)$ that has Fourier transform $\Pi(f)$ is bandlimited with bandwidth $b \geq \frac{1}{2}$. Another example is the signal with Fourier transform given by a **raised cosine**

$$\hat{x}(f) = \Pi(f)\big(1 + \cos(2\pi f)\big) = \begin{cases} 1 + \cos(2\pi f) & |f| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth $b \geq \frac{1}{2}$. The time domain signal is found by applying the inverse Fourier transform

$$x(t) = \text{sinc}(t) + \tfrac{1}{2}\,\text{sinc}(t+1) + \tfrac{1}{2}\,\text{sinc}(t-1). \qquad \text{(Exercise 5.7)}$$

Another example is the signal with Fourier transform given by the **triangle pulse**

$$\triangle(f) = \begin{cases} f + 1 & -1 < f < 0 \\ 1 - f & 0 \leq f < 1 \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth $b \geq 1$. The corresponding time domain signal is given by the square of the sinc function $\text{sinc}^2(t)$ (Exercises 5.2). These bandlimited signals and their Fourier transforms are plotted in Figure 5.7.

It happens that bandlimited signals are never finite. We can reasonably suppose that all signals ever encountered in practice are finite and so no signals encountered in practice are truely bandlimited. However, many practically occuring signals are approximately bandlimited, that is, their Fourier transform is small for frequencies larger than some positive number b. For example, in Test 7 the Fourier transform of an audio signal taken from a lecture recording is plotted (Figure 5.8). This signal appears approximately bandlimited with bandwidth a little larger than 8 kHz.

A surprising result is that every square integrable bandlimited signal x with bandwidth b can be written as a sum of time-scaled and time-shifted sinc functions, that is, in the form

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \,\text{sinc}(Ft - n) \tag{5.4.1}$$

where c is a square integrable complex valued sequence and $F = 2b$. This is a consequence of a property of the set of square integrable signals L_2 called

completeness [Rudin, 1986, Theorem 3.11 and page 91]. The sum on the right hand side of (5.4.1) is often called **sinc interpolation**. Evaluating the signal x at integer multiples of $P = \frac{1}{F}$ we find that

$$x(\ell P) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(\ell - n) = c_\ell$$

because $\operatorname{sinc}(\ell - n)$ is equal to 1 when $\ell = n$ and 0 otherwise. So, the elements of the sequence c correspond with samples of the signal x taken at integer multiples of $P = \frac{1}{F} = \frac{1}{2b}$, that is, $c_n = x(nP)$. The positive real number P is called the **sampling period** and its reciprical F the **sampling rate**. It follows that every square integrable bandlimited signal x with bandwidth b can be reconstructed from samples taken at rate $F = 2b$, that is,

$$x(t) = \sum_{n \in \mathbb{Z}} x(nP) \operatorname{sinc}(Ft - n).$$

This result known as the **Nyquist sampling theorem**. This motivated use of this recontruction method in Tests 1, 2, 3, and 5.

5.5 The discrete-time Fourier transform

Let x be a square integrable bandlimited signal with bandwidth b and let c be the square summable sequence containing samples of x at sampling rate $F = \frac{1}{P} = 2b$, that is, $c_n = x(nP)$. Suppose temporarily that c is also absolutely summable. From (5.4.1), the Fourier transform of x is

$$\begin{aligned}\hat{x}(f) = \mathcal{F}x(f) &= \mathcal{F}\left(\sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n)\right) \\ &= \sum_{n \in \mathbb{Z}} c_n \mathcal{F}(\operatorname{sinc}(Ft - n)) \qquad (5.5.1) \\ &= P\Pi(fP) \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi P n f} \\ &= P\Pi(fP)\mathcal{D}c(Pf) \qquad (5.5.2)\end{aligned}$$

where

$$\mathcal{D}c(f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f} \qquad (5.5.3)$$

is called the **discrete-time Fourier transform** of the sequence c. The interchange of Fourier transformation and summation on line (5.5.1) is justified by **Lebesgue's dominated convergence theorem** [Rudin, 1986, Section 1.34] (Exercise 5.17). We write $\hat{c} = \mathcal{D}c$ for the discrete-time Fourier transform of c. The discrete-time Fourier transform $\hat{c} = \mathcal{D}c$ is a periodic

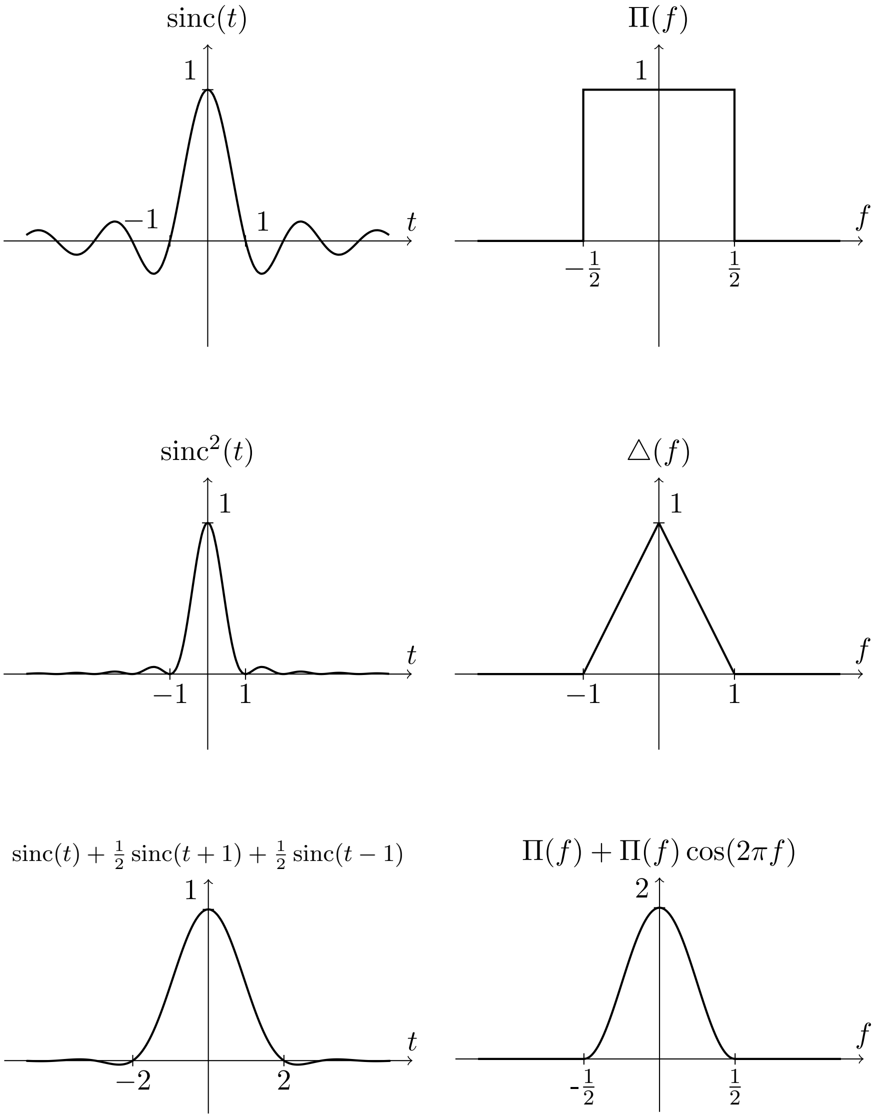

Figure 5.7: Bandlimited signals $\mathrm{sinc}(t)$, $\mathrm{sinc}^2(t)$, and $\mathrm{sinc}(t)+\frac{1}{2}\,\mathrm{sinc}(t+1)+\frac{1}{2}\,\mathrm{sinc}(t-1)$ with bandwidth $\frac{1}{2}$, 1, and $\frac{1}{2}$ respectively.

function from $\mathbb{R} \to \mathbb{C}$, that is, $\hat{c}$ is a periodic signal with period 1. For example, the discrete-time Fourier transform of the delta sequence is

$$\mathcal{D}\delta(f) = \sum_{n\in\mathbb{Z}} \delta_n e^{-j2\pi nf} = 1.$$

The sequence

$$\Pi(n/2) = \begin{cases} 1 & |n| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

has discrete-time Fourier transform

$$\mathcal{D}\big(\Pi(n/2)\big) = e^{-2\pi jf} + 1 + e^{2\pi jf} = 1 + 2\cos(2\pi f)$$

and the sequence $2^{-n}u_n$ has discrete-time Fourier transform (Exercise 5.14)

$$\mathcal{D}\big(2^{-n}u_n\big) = \sum_{n\in\mathbb{Z}} 2^{-n}u_n e^{-j2\pi nf} = \sum_{n=0}^{\infty} \frac{e^{-j2\pi nf}}{2^n} = \frac{2}{2 - e^{-j2\pi f}}.$$

These sequences and their discrete-time Fourier transforms are plotted in Figure 5.10.

We now use the discrete-time Fourier transform to compute and plot the Fourier transform of some sampled audio signals. Test 7 uses (5.5.2) to compute the Fourier transform of a 20 s segment of audio from a lecture recording. In Test 8 this audio signal is passed through the Butterworth filter constructed in Test 6 and the Fourier transform of the response is plotted.

Test 7 (**The Fourier transform of a lecture recording**)

In this test we consider a 20 s segment of audio taken from the lecture video `ch1sec3.mp4`. This 34.8 MB file contains both compressed video (H.264 codec) and audio (mp3 codec) of duration 23 min and 36 s. The audio is mono and sampled at rate $F = 22\,050$ Hz. The `ffmpeg` program is used to extract a 20 s segment of audio starting at time 85 s and ending at time 105 s. The segment is decompressed to wav format. The command used is:

```
ffmpeg -i ch1sec3.mp4 -ss 85 -t 20 audio.wav
```

The resulting file `audio.wav` is 882 kB in size and contains $N = 440998$ samples that we denote by $c_0, c_1, \ldots, c_{N-1}$. Each sample takes a value in the interval $[-1, 1]$. We put $c_n = 0$ when $n < 0$ or $n \geq N$. The reconstructed audio signal is given by

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n).$$

From (5.5.2) the Fourier transform of this signal is $\hat{x}(f) = P\Pi(Pf)\hat{c}(Pf)$ where

$$\hat{c}(f) = \mathcal{D}c(f) = \sum_{n\in\mathbb{Z}} c_n e^{-j2\pi nf} = \sum_{n=0}^{N-1} c_n e^{-j2\pi nf} \tag{5.5.4}$$

is the discrete-time Fourier transform of the sequence of samples. Figure 5.8 shows a plot of the magnitude of the Fourier transform for frequencies in the interval -12 kHz to 12 kHz. The plot is constructed by evaluting $|\hat{c}(f)|$ at all $K = 1201$ frequencies

$$f_k = -12000 + 20k \qquad k = 0, \dots, K-1,$$

that is, from -12 kHz to 12 kHz in steps of 20 Hz. It takes approximately 137 s to compute the Fourier transform at all of these frequencies. Evaluating the Fourier transform at a particular frequency requires calculating and accumlating each of the N terms in the sum (5.5.4). We hypothesise it to take approximately

$$\frac{137\,\mathrm{s}}{NK} \approx 260\,\mathrm{ns}$$

to compute each term. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the `Scala` programming language.

The audio recording contains human voice that primarily resides at lower frequencies below 4 kHz. Audible in the recording is a faint high pitched hum. The cause of this is unknown. It might be a feature of the (probably low quality) webcam microphone used to record the audio. This hum is represented in Figure 5.8 by the spikes occurring at approximately ± 8 kHz and also by the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated. Figure 5.9 is a plot of the Fourier transform for frequencies from 7998 Hz to 8002 Hz in steps of 5 mHz. This gives a high resolution view of the spike that occurs near 8 kHz. The magnitude of the Fourier transform is precisely zero for frequencies $|f| > F/2 = 11\,025$ Hz due to $\Pi(Pf)$ occurring in the definition of $\hat{x}$. However, in Figure 5.8 it is apparent that the Fourier transform is small if $|f|$ is a little larger than 8 kHz. This audio signal appears approximately bandlimited with bandwidth a little larger 8 kHz.

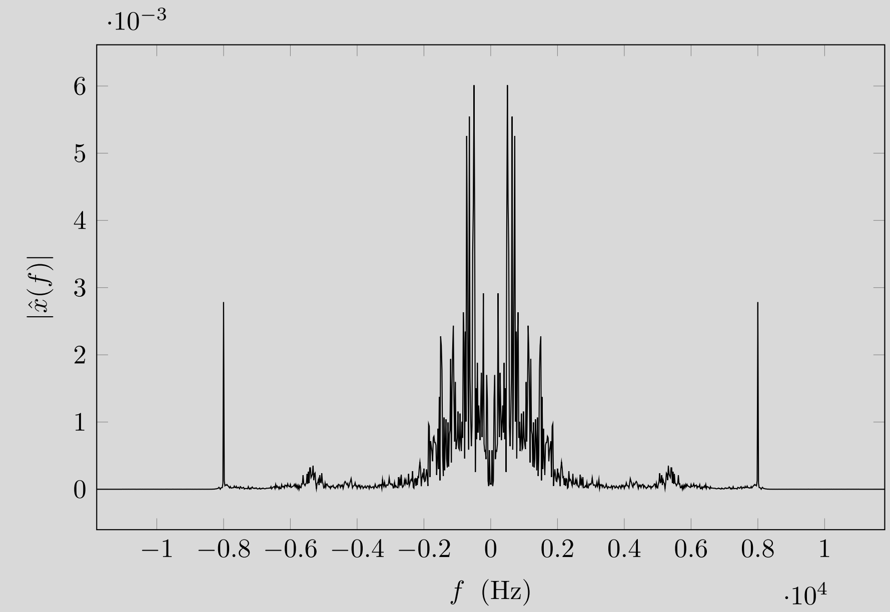

Figure 5.8: Magnitude of the Fourier transform of 20 s of audio from a lecture recording. The human voice signal is primarily contained in the low frequency region below 5 kHz. The spikes occurring at approximately ±8 kHz and the region between 4900 Hz and 5900 Hz where the magnitude is elevated are audible in the recording as a high pitched hum.

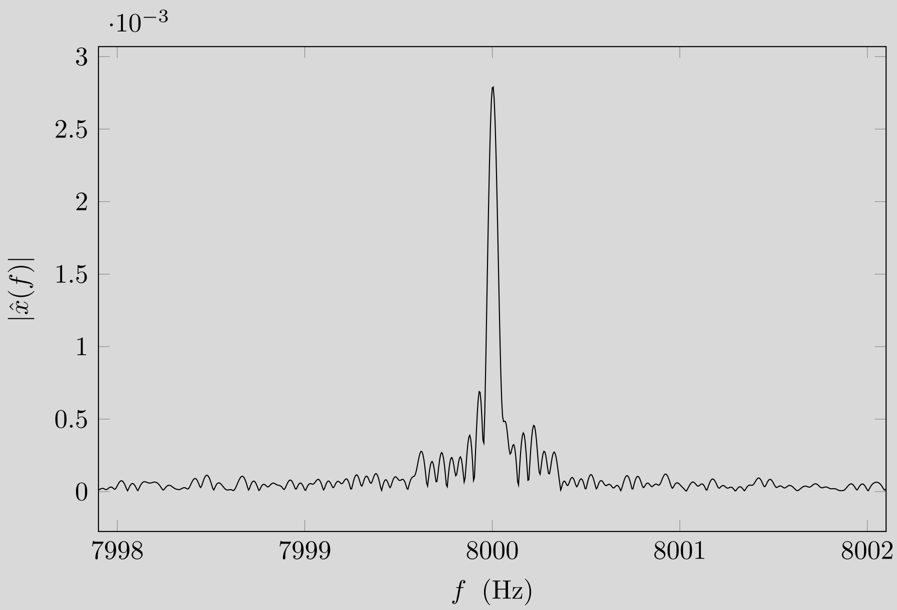

Figure 5.9: A plot of the magnitude of the Fourier transform zoomed in on the spike at 8 kHz.

Test 8 (**Butterworth filtered lecture recording**)
We consider again the 20 s audio signal from Test 7. In this test we pass this signal through the second order Butterworth filter from Test 6 with cuttoff frequency approximately 3041 Hz. The output of the Butterworth filter is fed back to the soundcard input and recorded at 22 050 Hz. The recorded samples are written to the file `filtered.wav`. Listening to `filtered.wav` confirms that the high pitched hum is weaker than it is in the original audio signal. The Fourier transform of the Butterworth filtered signal is plotted in Figure 5.10. This figure is constructed by the same procedure as used for Figure 5.8 from Test 7. Observe that the spikes occurring at approximately ±8 kHz are less prominent than in Figure 5.8.

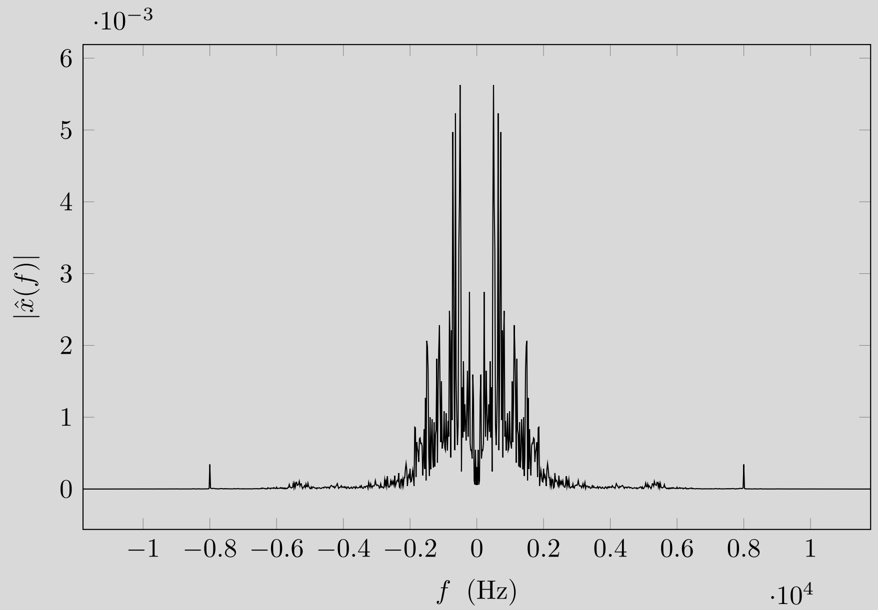

Figure 5.10: Magnitude of the Fourier transform of 20 s of audio from Test 7 after being passed through the second order Butterworth filter from Test 6 with cutoff frequency approximately 3041 Hz. The magnitude of the Fourier transform at higher frequencies is attenuated when compared with the Fourier transform of the original audio signal (Figure 5.8). In particular, the spikes occurring at approximately ±8 kHz are less prominent than in Figure 5.8. The high pitched hum that is audible in the original audio signal is significantly weaker in the Butterworth filtered audio signal.

Under our assumption that c is absolutely summable

$$|\hat{c}(f)| = |\mathcal{D}c(f)| \leq \sum_{n\in\mathbb{Z}} \left| c_n e^{-j2\pi n f} \right| = \sum_{n\in\mathbb{Z}} |c_n| = \|c\|_1 < \infty$$

and so the discrete Fourier transform $\mathcal{D}c(f)$ is finite for all f. A discrete-time Fourier transform can also be assigned to sequences that are only square summable and not necessarily absolutely summable. In this case one interprets the sum in (5.5.3) as

$$\hat{c}(f) = \mathcal{D}c(f) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{-j2\pi n f}.$$

This is analogous to how the Fourier transform of a square integrable signal was assigned using the Cauchy principal value (5.1.2). The relationship (5.5.2) between $\hat{x}$ and $\hat{c} = \mathcal{D}c$ still holds in this case provided that equality is weakened to equality almost everywhere (Exercise 5.18).

Equation (5.5.2) relates the Fourier transform of the bandlimited signal x to the discrete-time Fourier transform of its sequence of samples c. The sequence of samples $c_n = x(nP)$ can be recovered by evaluating the inverse Fourier transform

$$\begin{aligned} c_n = x(nP) &= \mathcal{F}^{-1}\hat{x}(nP) \\ &= \int_{-\infty}^{\infty} P\Pi(Pf)\mathcal{D}c(Pf)e^{j2\pi f n P} df \\ &= \int_{-\infty}^{\infty} \Pi(\gamma)\hat{c}(\gamma)e^{j2\pi\gamma n} d\gamma \qquad \text{(change variable } \gamma = fP) \\ &= \int_{-1/2}^{1/2} \hat{c}(\gamma)e^{j2\pi\gamma n} d\gamma. \end{aligned}$$

We obtain the following relationship between the square integrable sequence c and its periodic discrete-time Fourier transform $\hat{c} = \mathcal{D}c$,

$$c_n = \int_{-1/2}^{1/2} \hat{c}(f)e^{j2\pi f n} df.$$

The right hand side of this expression is called the **inverse discrete-time Fourier transform**. The element c_{-n} is also known as the nth **Fourier coefficient** of the periodic function $\hat{c}$.

Replacing Pf with f in (5.5.2) and multiplying by F we obtain

$$\Pi(f)\hat{c}(f) = F\hat{x}(Ff).$$

Because $\hat{c}(f)$ has period 1 and because the rectangle function $\Pi(f)$ takes the value 1 on the interval $(-\frac{1}{2}, \frac{1}{2})$ the product $\Pi(f)\hat{c}(f)$ corresponds with

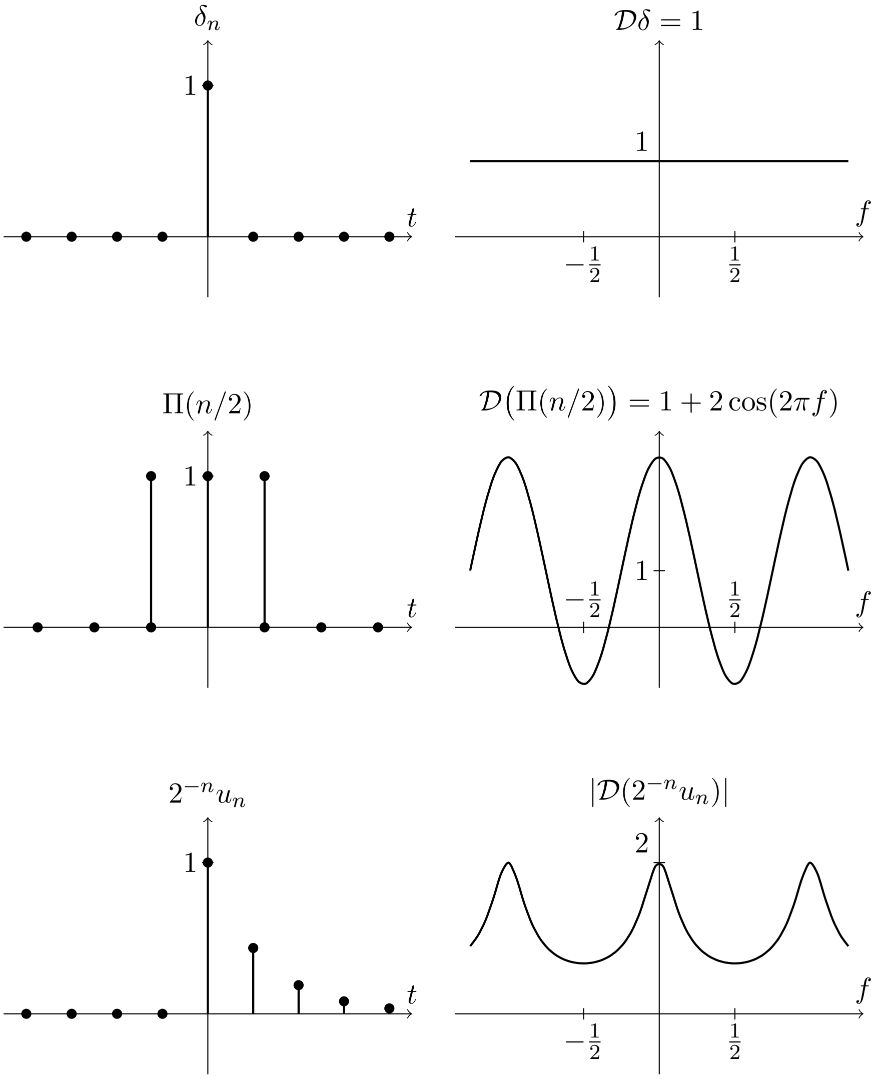

Figure 5.10: The sequences δ and $\Pi(n/2)$ and their discrete-time Fourier transforms (top and middle). The sequence $2^{-n}u_n$ and the magnitude of its discrete-time Fourier transform $4/(5 - 4\cos(2\pi f))$ (bottom). The discrete-time Fourier transforms are periodic signals with period 1.

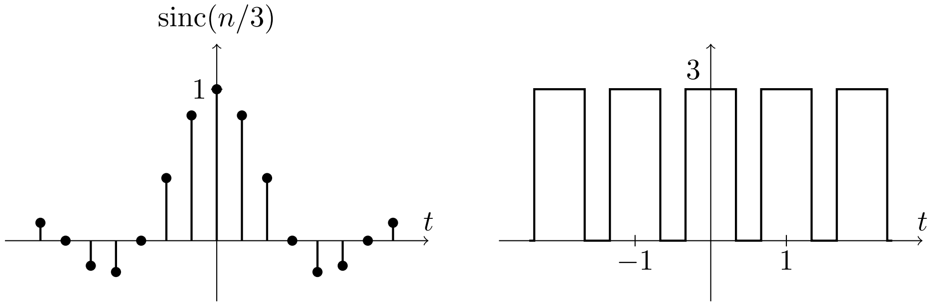

Figure 5.11: The sequence $\operatorname{sinc}(n/3)$ and its discrete-time Fourier transform $3\sum_{m\in\mathbb{Z}}\Pi(3f+3m)$.

a single period of $\hat{c}$ centered at the origin. This suggests reconstructing $\hat{c}$ by summing shifts of $\hat{x}(Ff)$ by integers, that is,

$$\hat{c}(f) = \mathcal{D}c(f) = F\sum_{m\in\mathbb{Z}}\hat{x}(Ff+Fm) \qquad \text{a.e.} \tag{5.5.5}$$

The right hand side is called a **periodic summation** of $\hat{x}$. Examples of periodic summations are plotted in Figure 5.12. The relationship (5.5.5) provides a convenient means of determining the discrete-time Fourier transform of a sequence by considering it as the samples of a signal x for which the Fourier transform is already known. For example, consider the sequence $c_n = \operatorname{sinc}(n/3)$ given by sampling the sinc function at rate $F = 3$. Computing the discrete-time Fourier transform of c directly using (5.5.3) is not straightforward. However, we know that $\mathcal{F}\operatorname{sinc}(f) = \Pi(f)$ and so (5.5.5) gives $\hat{c}(f) = 3\sum_{m\in\mathbb{Z}}\Pi(3f+3m)$. The sequence $\operatorname{sinc}(n/3)$ and its discrete-time Fourier transform are plotted in Figure 5.11. We will find the relationship (5.5.5) particularly convient for the design of digital filters in Section 6.4.

Substituting Pf for f in (5.5.5) and multiplying by P we find that

$$P\hat{c}(Pf) = \sum_{m\in\mathbb{Z}}\hat{x}(f+Fm) \qquad \text{a.e.}$$

In the case that x is bandlimited with bandwidth $b < F/2$ the Fourier transform $\hat{x}$ can be recovered by multiplying both sides of this equation by the rectangle function $\Pi(Pf)$. Equation (5.5.2) is reobtained in this way. This effect can be seen in the top plot in Figure 5.5.2 whereby multiplication of the periodic summation on the right by $\Pi(f)$ recovers the rectangular pulse centered and the origin. Recovery of $\hat{x}$ in this way is not possible if x has bandwidth larger than $F/2$. Consider, for example, the middle and bottom plots in Figure 5.12 where multiplication of the periodic summation on the right by $\Pi(f)$ would not recover the original signals on the left.

While the relationship (5.5.2) requires x to be bandlimited with bandwidth $b < F/2$, the relationship (5.5.5) between the discrete-time Fourier transform $\hat{c}$ and the periodic summation of $\hat{x}$ can still hold even when x is not bandlimited and, in this case, the sequence c does still correspond with samples of the signal x at period P. To see this, care must first be taken to ensure that the periodic summation of $\hat{x}$ is a well defined signal. This will be the case if $\hat{x}$ is absolutely integrable (Exercise 5.19). In this case, applying the inverse discrete-time Fourier transform to both sides of (5.5.5) gives

$$c_n = \int_{-1/2}^{1/2} F \sum_{m\in\mathbb{Z}} \hat{x}(Ff + Fm) e^{j2\pi fn} df.$$

Under our assumption that $\hat{x}$ is absolutely integrable the dominated convergence theorem can be used to justify exchanging infinite summation and integration [Pinksy, 2002, Section 4.2] and so

$$c_n = F \sum_{m\in\mathbb{Z}} \int_{-1/2}^{1/2} \hat{x}(Ff + Fm) e^{j2\pi fn} df.$$

By the change of variables $\gamma = F(f + m)$,

$$c_n = \sum_{m\in\mathbb{Z}} \int_{Fm-F/2}^{Fm+F/2} \hat{x}(\gamma) e^{j2\pi\gamma n/F} e^{-j2\pi mn} d\gamma.$$

The term $e^{-j2\pi mn} = 1$ because both m and n are integers and the sum of integrals can be combined into a single integral over the entire real line leading to

$$c_n = \int_{-\infty}^{\infty} \hat{x}(\gamma) e^{j2\pi\gamma n/F} d\gamma = \mathcal{F}^{-1}\hat{x}(n/F) = x(Pn)$$

where the integral is identified as the inverse Fourier transform of $\hat{x}$ (5.1.1). We have found that the sequence c with discrete-time Fourier transform $F\sum_{n\in\mathbb{Z}} \hat{x}(Ff+Fm)$ is precisely the samples of the signal x at period $P = \frac{1}{F}$. This does not require that x be bandlimited.

The reconstructed bandlimited signal $\sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(Ft - n)$ will not be equal to x if x has bandwidth $b > F/2$. The reconstructed signal will have Fourier transform $\Pi(Pf)\sum_{m\in\mathbb{Z}} \hat{x}(f + mF) = P\hat{c}(Pf)$ and this will only be equal to $\hat{x}$ in the case that x has bandwidth $b < F/2$. Figure 5.13 shows the affect of reconstructing the signal $x(t) = e^{-\pi F^2 f^2/2}$ from samples taken at rate $F = \frac{1}{P}$. The Fourier transform of x can be shown to be $\hat{x}(f) = \sqrt{2}Pe^{-2\pi P^2 f^2}$. This signal is not bandlimited. The samples $c_n = x(nP)$ and corresponding discrete-time Fourier transform $\hat{c}$ are shown in the middle plot. Observe that the translates of the Fourier transform $\hat{x}$ overlap the translate at the origin causing the Fourier transform of the reconstructed signal (bottom of Figure 5.13) to differ from that of the original signal. This phenomenon is referred to as **aliasing**.

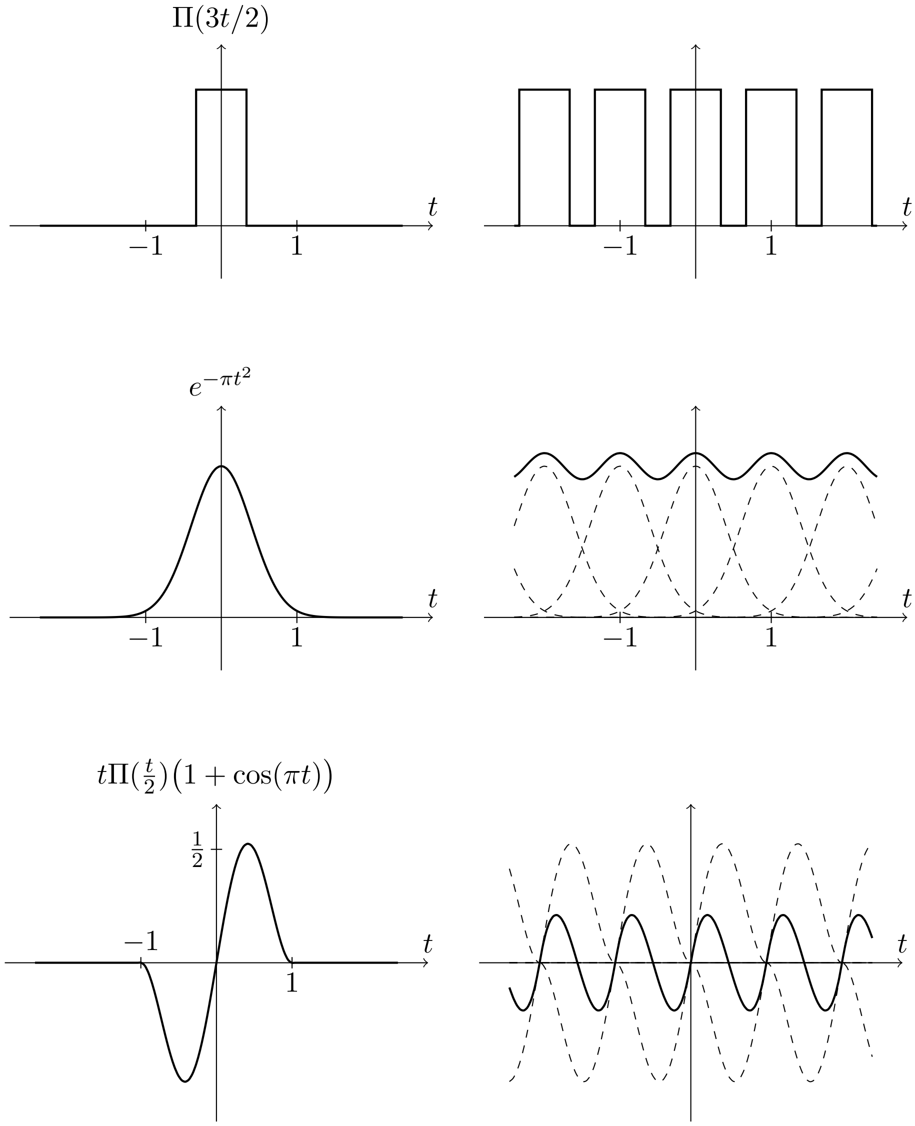

Figure 5.12: Signals x and their periodic summations $\sum_{m\in\mathbb{Z}} x(t+m)$. The periodic summation is shown by the solid lines on the left. The dashed lines are the individual translations being summed.

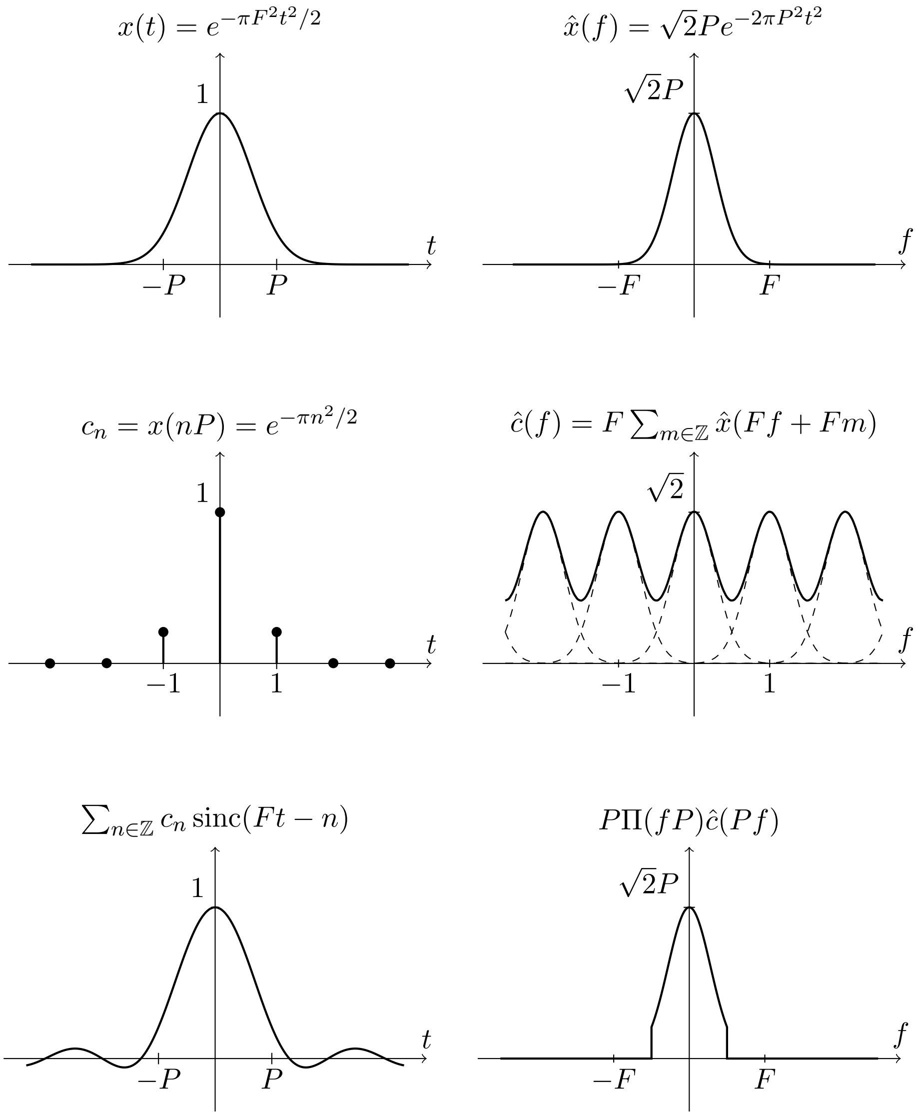

Figure 5.13: Top: the signal $x(t) = e^{-\pi F^2 t^2/2}$ and its Fourier transform $\hat{x}(f) = \sqrt{2} P e^{-2\pi P^2 t^2}$. Middle: the sequence of samples $c_n = x(nP)$ and discrete-time Fourier transform $\hat{c}$ related to the periodic summation of $\hat{x}$. Bottom: The bandlimited signal reconstructed from the sequence c and its Fourier transform. Observe that the reconstructed bandlimited signal is not equal to the original signal x. The translates of the Fourier transform $\hat{x}$ overlap the translate at the origin, a phenomenon referred to as **aliasing**.

5.6 The fast Fourier transform

In Test 7 the Fourier transform of a 20 s audio signal consisting of $N = 440998$ consecutive samples was evaluated. This scenario where only a finite number, say N, of consecutive samples of a signal is available is common in practice. Let c be a sequence with elements $c_0, c_1, \dots, c_{N-1}$ equal to the N samples. A convenient assumption is that the remaining samples are equal to zero, that is, $c_n = 0$ when $n < 0$ or $n \geq N$. This assumption was made in Test 7.

With this assumption the discrete-time Fourier transform of the sequence c is given by the finite sum

$$\hat{c}(f) = \mathcal{D}c(f) = \sum_{n\in\mathbb{Z}} c_n e^{-j2\pi nf} = \sum_{n=0}^{N-1} c_n e^{-j2\pi nf}.$$

The values of $\hat{c}(f)$ for f a multiple of $\frac{1}{N}$ have a number of convenient properties. Denote these values by

$$(\mathcal{D}_N c)_k = \mathcal{D}c\big(\tfrac{k}{N}\big) = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/N} \qquad k \in \mathbb{Z}. \tag{5.6.1}$$

This is called the **discrete Fourier transform** (as opposed to the discrete *time* Fourier transform). The discrete Fourier transform $\mathcal{D}_N c$ is a sequence with elements given by the discrete-time Fourier transform $\mathcal{D}c$ evaluated at multiples of $\frac{1}{N}$. As usual we include write $\mathcal{D}_N c_k$ to denote the value of $\mathcal{D}_N c$ at $k \in \mathbb{Z}$. We will regularly include brackets and use $(\mathcal{D}_N c)_k$ to denote this value. The positive integer N is called the **length** of the transform. In practical applications N often corresponds with the number of samples of a signal that have been obtained.

The discrete Fourier transform is a periodic sequence with period N as a result of the discrete-time Fourier transform $\hat{c} = \mathcal{D}c$ having period 1, that is,

$$(\mathcal{D}_N c)_k = \hat{c}\left(\frac{k}{N}\right) = \hat{c}\left(\frac{k + mN}{N}\right) = (\mathcal{D}_N c)_{k+mN} \quad \text{for all } k, m \in \mathbb{Z}.$$

Because of this it is sufficient to know only $(\mathcal{D}_N c)k$ for $k = 0, \dots, N-1$ in order to know the entire sequence $\mathcal{D}_N c$. Given $d = \mathcal{D}_N c$ the original samples $c_0, \dots, c_{N-1}$ can be recovered by

$$c_n = (\mathcal{D}_N^{-1} d)_n = \frac{1}{N} \sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} \qquad n = 0, \dots, N-1.$$

This is called the **inverse discrete Fourier transform** (Excersise 5.12). Taking complex conjugates on both sides gives

$$c_n^* = \frac{1}{N} \sum_{k=0}^{N-1} d_k^* e^{-j2\pi nk/N} = \frac{1}{N} (\mathcal{D}_N d^*)_n \qquad n = 0, \dots, N-1.$$

A practical consequence of this is that the inverse discrete Fourier transform can be evaluted by applying the complex conjugate, taking the discrete Fourier transform, applying the complex conjugate again, and finally dividing by N. That is, if d is a sequence, then $\mathcal{D}_N^{-1} d = \frac{1}{N}\mathcal{D}_N^* d^*$.

Suppose that we wish to evaluate the discrete Fourier transform $\mathcal{D}_N c$ of the 20 s audio signal comprising of $N = 440998$ samples from Test 7. In Test 7 we hypothesised that approximately 260 ns are required to compute each term in the sum (5.5.4). We require to compute the sum for each $k = 0, \dots, N-1$ and so we might expect that

$$N^2 \times 260\,\text{ns} \approx 50\,565\,\text{s} \approx 14 \text{ hours} \tag{5.6.2}$$

will be required to compute $\mathcal{D}_N c$ for this 20 s audio signal! A primary cause of this lengthy computation time is the quadratic term N^2 that occurs in the expression above. The amount of time required grows proportionally with the square of the length of the transform. Suppose that instead of 20 s of audio we have 1 hour and $N = 60 \times 60 \times 22050 = 79380000$ samples. The amount of time required in this case is approximated by $N^2 \times 260\,\text{ns} \approx 52$ years!

Computing the discrete Fourier transform by direct application of the formula (5.6.1) is too slow when N is large. Fortunately, much faster algorithms exist. The algorithms are appropriately called **fast Fourier transforms**. The specific algorithm used depends on N. The simplest case is when $N = 2^m$ is a power of 2. In this case an algorithm attributed to Cooley and Tukey [1965] can be used. When $N = 2^m$ is divisible by 2 the sum in (5.6.1) can be split into two parts corresponding with n being even or odd,

$$(\mathcal{D}_N c)_k = \sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} + \sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N}. \tag{5.6.3}$$

Put $M = N/2$ and let p be the sequence with elements $p_n = c_{2n}$, that is, the elements of p are the even indexed elements of c. Now the first term in (5.6.3) can be written in the form

$$\sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} = \sum_{n=0}^{M-1} p_n e^{-j2\pi nk/M} = (\mathcal{D}_M p)_k,$$

that is, this term is the discrete Fourier transform of length $M = N/2$ of the sequence p. Let q be the sequence with elements $q_n = c_{2n+1}$, that is, q contains the odd indexed elements of c. The second term in (5.6.3) can be written in the form

$$\begin{aligned}\sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N} &= e^{-j2\pi k/N} \sum_{n=0}^{M-1} q_n e^{-j2\pi nk/M} \\ &= e^{-j2\pi k/N} (\mathcal{D}_M q)_k,\end{aligned}$$

that is, this term is the discrete Fourier transform of length $M = N/2$ of the sequence q multiplied by $e^{-j2\pi k/N}$. Combining these results we have

$$(\mathcal{D}_N c)_k = (\mathcal{D}_{N/2} p)_k + e^{-j2\pi k/N}(\mathcal{D}_{N/2} q)_k.$$

We see that the discrete Fourier transform $\mathcal{D}_N c$ can be evaluated by computing two smaller discrete Fourier transforms $\mathcal{D}_{N/2} p$ and $\mathcal{D}_{N/2} q$ of length $N/2$. Both of these smaller transforms are sequences that are periodic with period $N/2$ and so it is sufficient to know their values only for $k = 0, \ldots, \frac{N}{2} - 1$. These $N/2$ length transforms can in turn be computed by two transforms of length $N/4$ and so on until transforms of length 1 are obtained. In this case $(\mathcal{D}_1 c)_k = \sum_{n=0}^{0} c_n e^{-j2\pi nk} = c_0$ for all $k \in \mathbb{Z}$.

The computational cost of this procedure can be analysed as follows. Suppose that C_N is the number of complex arithmetic operations (complex additions and multiplications) required to compute the discrete Fourier transform $\mathcal{D}_N c$ of length $N = 2^m$. The computation requires calculation of two transforms of length $N/2$ followed by N complex multiplications and N additions. The multiplications arise from the multiplication of $(\mathcal{D}_{N/2} q)_k$ by $e^{-j2\pi k/N}$ and the additions arise from summing the result of this product with $(\mathcal{D}_{N/2} p)_k$. The number of operations satisfies

$$C_N = 2C_{N/2} + 2N \qquad N \geq 2.$$

Because $(\mathcal{D}_1 c)_k = c_0$ we have $C_1 = 0$, that is, computing a discrete Fourier transform of length 1 requires no complex operations at all. Putting $a_m = C_{2^m}$ we have

$$a_0 = C_1 = 0 \qquad a_m = 2a_{m-1} + 2^{m+1} \quad m \geq 1. \tag{5.6.4}$$

This type of recursive equation is called a **difference equation** and will be studied further in Section 6.6. Exercise 6.6 shows that

$$C_N = a_m = 2^{m+1} m = 2N \log_2 N.$$

Observe that the number of operations (and hence the amount of time required) grows proportionally to $N \log_2 N$ rather than N^2. Suppose that each complex operation requires no more than 260 ns. For the 20 s audio signal consisting of $N = 440998$ samples the amount of time required will be less than

$$2N \log_2 N \times 260\,\text{ns} \approx 4.3\,\text{s}. \tag{5.6.5}$$

This is more reasonable than 14 hours! If instead we have 1 hour of audio and $N = 79380000$ samples the amount of time required is hypothesised to be less than $1084\,\text{s} \approx 18\,\text{min}$. This is very reasonable when compared with the 52 years hypothesised to be required by direct application of formula (5.6.1). In practice the computation time will vary based on the computer used and the specific algorithm implementation. Nevertheless, these numbers indicate that a fast Fourier transform of large length can be computed within a reasonable amount of time. This is not possible by direct application of formula (5.6.1). Test 9 compares the practical running time of various discrete Fourier transform implementations.

In the above computation of run times we have neglected that the fast Fourier transform we have described required the length N to be a power of two. Other algorithms exist for the case when N is not a power of two [Rader, 1968; Bluestein, 1968; Frigo and Johnson, 2005]. These algorithms deliver similarly dramatic computational savings. Even so, the restriction of the length to a power of 2 is often not a significant drawback in practical applications. Consider again the example from Test 7 with $N = 440998$ samples. Denote by

$$L = 2^{\lceil \log_2 N \rceil} = 2^{19} = 524288$$

the smallest power of 2 greater than N. We can use the fast Fourier transform algorithm described to compute the discrete Fourier transform $D_L c$ of length L. This transform is a sequence with period L and elements

$$(\mathcal{D}_L c)_k = \hat{c}\left(\frac{k}{L}\right) = \sum_{n=0}^{L-1} c_n e^{-j2\pi nk/L} = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/L} \qquad k \in \mathbb{Z}.$$

The second sum follows from our assumption that $c_n = 0$ for $n \geq N$. The elements of $\mathcal{D}_L c$ are the values of the discrete-time Fourier transform $\hat{c}$ at multiples of $\frac{1}{L}$ rather than $\frac{1}{N}$. This fact is often of no significant consequence and can even be of benefit for some applications [Quinn and Hannan, 2001; Quinn et al., 2008]. The original samples $c_0, \dots, c_{N-1}$ can still be recovered by application of the inverse transform of length L, that is,

$$c_n = (\mathcal{D}_L^{-1} \mathcal{D}_L c)_n \qquad n = 0, \dots, N-1.$$

This procedure of increasing the length of the transform is often called **zero padding** on account of the fact that the samples $c_N, c_{N+1}, \dots, c_{L-1}$ are assumed to be zero. Test 10 presents a practical example of zero padding for the purpose of filtering the 20 s audio recording from Test 7.

Test 9 (**Benchmarking the fast Fourier transform**)

In this test we compare the computational complexity of practical implementations of the discrete Fourier transform. Three differenent implementations are compared: a direct implementation by formula (5.6.1), an implementation of the fast Fourier transform of Cooley and Tukey [1965] when the length $N = 2^m$ is a power of 2 as described in Section 5.6, and an implementation from an optimised fast Fourier transform library called

`JTransforms`. The `JTransforms` library contains implementations of fast Fourier transforms of all lengths, not just powers of 2.

Figure 5.14 shows the run-time in seconds versus transform length. For the `JTransforms` library and formula (5.6.1) the length of the transforms is given by the sequence $N_k = \lceil 2^{6+k/2} \rceil$ for $k = 0, 1, 2, \dots$. For our implementation of the Cooley and Tukey [1965] algorithm the length must be a power of two and is given by $N_k = 2^{6+k/2}$ for $k = 0, 2, 4, \dots$. The dashed lines indicate the approximate running times given by (5.6.2) and by (5.6.5). These approximations appear reasonably accurate on the log scale used in Figure 5.14. The fast Fourier transform algorithms are considerably faster than formula (5.6.1) as expected. For example, when the length is $N = 2^{21} = 2097152$ the `JTransforms` library required approximately 0.58 s whereas formula (5.6.1) is hypothesised by (5.6.2) to require approximately $N^2 \times 260\,\text{ns} \approx 13$ days.

The optimised algorithms from the `JTransforms` library are considerably faster than our implementation of the Cooley and Tukey [1965] algorithm. Observe the jagged nature of the run-time with the `JTransforms` library. The algorithms used by the library for length N_k and odd k appear slower than when k is even so that the length is a power of 2. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the `Scala` programming language.

Test 10 (Filtering a lecture recording by fast Fourier transform)

We again consider the 20 s segment of audio consisting of $N = 440998$ samples from Test 7. As in Test 7 we let c be the sequence with elements $c_0, \dots, c_{N-1}$ equal to the audio samples and put $c_n = 0$ for $n < 0$ or $n \geq N$. The reconstructed audio signal is given by

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n)$$

where $P = \frac{1}{F}$ is the sample period and $F = 22\,050\,\text{Hz}$ is the sample rate. The Fourier transform of x is $\hat{x} = \mathcal{F}x = P\Pi(Pf)\hat{c}(Pf)$ where $\hat{c} = \mathcal{D}c$. Audible in the recording is a faint high pitched hum. This hum appears in the Fourier transform as spikes occurring at $\pm 8\,\text{kHz}$ and also as the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated (Figure 5.8).

In this test we use a fast Fourier transform to remove this hum from the audio while minimally affecting the human voice. To do this we compute an

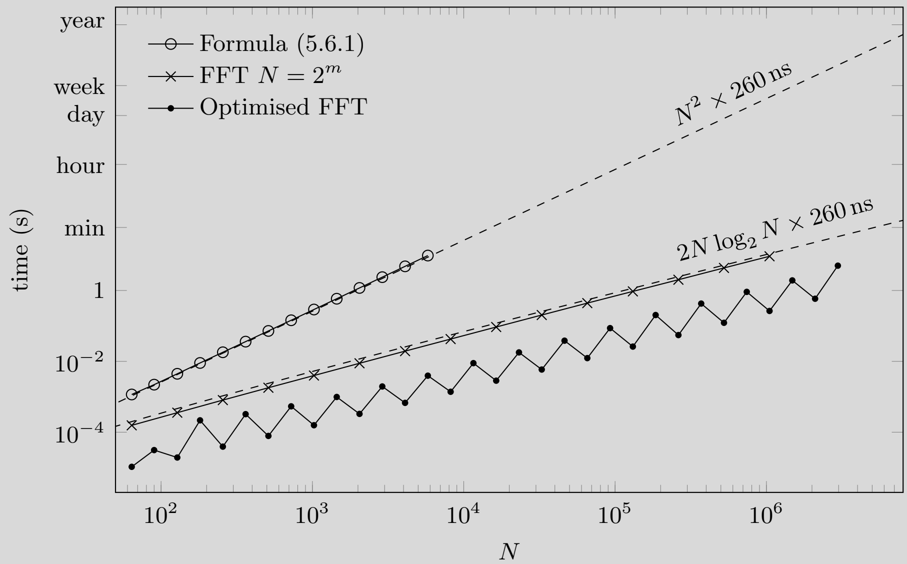

Figure 5.14: Comparison between run-times of the discrete Fourier transform computed directly by formula (5.6.1), by an implementation of the fast Fourier transform (FFT) of Cooley and Tukey [1965] described in Section 5.6, and by the optimised `JTransforms` fast Fourier transform library.

approximation of the bandlimited signal y with Fourier transform

$$\hat{y}(f) = \mathcal{F}y(f) = \begin{cases} 0 & |f| > 7200 \\ 0 & |f - 5400| < 500 \\ 0 & |f + 5400| < 500 \\ \hat{x}(f) & \text{otherwise.} \end{cases}$$

That is, y is the signal with Fourier transform equal to $\hat{x}$ except for those frequencies between 4900 Hz and 5900 Hz and above 7200 Hz where the Fourier transform is zero. Because y is bandlimited with bandwidth less than F,

$$y(t) = \sum_{n \in \mathbb{Z}} b_n \operatorname{sinc}(Ft - n)$$

where b is the sequence with elements $b_n = y(nP)$ given by samples of y at sample period P. Now $\hat{y} = P\Pi(Pf)\hat{b}(Pf)$ where $\hat{b} = \mathcal{D}b$ is the discrete-time Fourier transform of b. For f inside the interval $[-\frac{1}{2}, \frac{1}{2})$, the discrete-time

Fourier transforms $\hat{c}$ and $\hat{b}$ are related by

$$\hat{b}(f) = \begin{cases} 0 & |f| > 7200P \\ 0 & |f - 5400P| < 500P \\ 0 & |f + 5400P| < 500P \\ \hat{c}(f) & \text{otherwise.} \end{cases}$$

For $f \notin [-\frac{1}{2}, \frac{1}{2})$ a similar relationship can be obtained by appealing to the periodicity of $\hat{b}$ and $\hat{c}$. This is easiest to express by introducing the notation $\langle a \rangle = a - \lceil a \rfloor$ called the **centered fractional part** of $a \in \mathbb{R}$. Now

$$\hat{b}(f) = \begin{cases} 0 & |\langle f \rangle| > 7200P \\ 0 & |\langle f - 5400P \rangle| < 500P \\ 0 & |\langle f + 5400P \rangle| < 500P \\ \hat{c}(f) & \text{otherwise} \end{cases}$$

for all $f \in \mathbb{R}$.

Let $L = 2^{19} = 524288$ be the smallest power of 2 less than or equal to N. Using the fast Fourier transform we compute the discrete Fourier transform $\mathcal{D}_L c$ of length L of the sequence c. This yields values of $\hat{c}$ at multiples of $\frac{1}{L}$, that is,

$$(\mathcal{D}_L c)_k = \hat{c}\big(\tfrac{k}{L}\big) \qquad k \in \mathbb{Z}.$$

Let d be the sequence with elements

$$d_k = \mathcal{D}b(k/L) = \begin{cases} 0 & \big|\big\langle \frac{k}{L} \big\rangle\big| > 7200P \\ 0 & \big|\big\langle \frac{k}{L} - 5400P \big\rangle\big| < 500P \\ 0 & \big|\big\langle \frac{k}{L} + 5400P \big\rangle\big| < 500P \\ (\mathcal{D}_L c)_k = \hat{c}(k/L) & \text{otherwise.} \end{cases}$$

We do not necessarily have $d = \mathcal{D}_L b$ because b_n is not necessarily equal to zero for $n < 0$ and $n \geq L$. Nevertheless, we will suppose that $d \approx \mathcal{D}_L b$. In this case, application of the inverse discrete Fourier transform yeilds the periodic sequence $\tilde{b} = \mathcal{D}_L^{-1} d$ and we expect the first L elements of $\tilde{b}$ to be an approximation of the first L elements of b, that is,

$$\tilde{b}_n \approx b_n = y(nP) \qquad \text{for } n = 0, \dots, L-1.$$

An approximation of the signal y is now given by

$$y(t) \approx \tilde{y}(t) = \sum_{n=0}^{N-1} \tilde{b}_n \operatorname{sinc}(Ft - \ell).$$

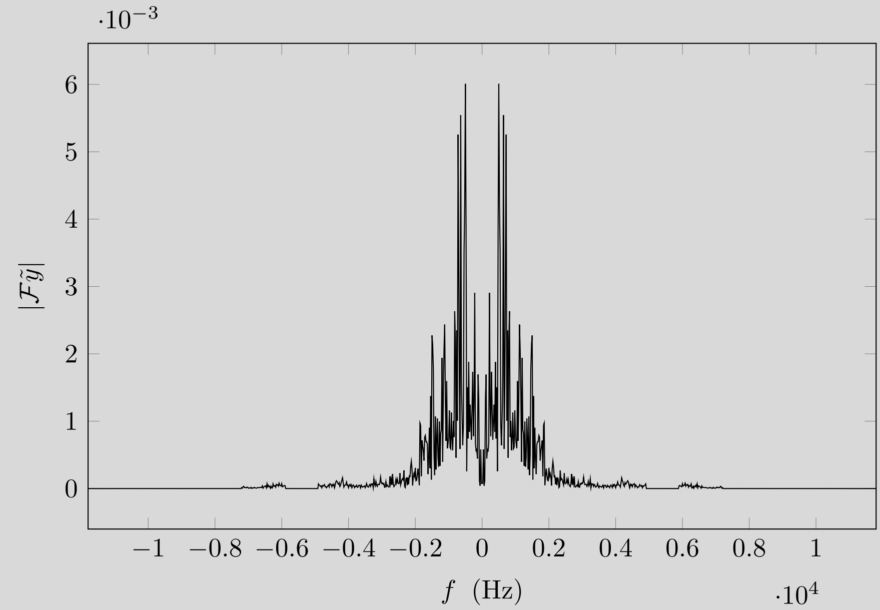

Figure 5.15: Plot of the magnitude of the Fourier transform $\mathcal{F}\tilde{y}$. The plot looks similar to that of the magnitude of the Fourier transform of the original audio signal (Figure 5.8) except that the spikes at ± 8 kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist.

Figure 5.15 plots the magnitude of Fourier transform $\mathcal{F}\tilde{y}$. Observe that $|\mathcal{F}\tilde{y}|$ looks similar to the magnitude of the Fourier transform of the original audio signal x plotted in Figure 5.8 except that the spikes at ± 8 kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist. The samples $\tilde{b}_0, \dots, \tilde{b}_{N-1}$ are written to the audio file `nohum.wav`. Listening to the audio confirms that the human voice signal remains, but the high pitched hum is no longer audible.

Exercises

5.1. Plot the signal $e^{-\alpha|t|}$ where $\alpha > 0$ and find its Fourier transform.

5.2. Plot the signal

$$\triangle(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

*5.3. Show that the sinc function is square integrable, but not absolutely integrable.

*5.4. Show the the magnitude spectrum of the normalised Butterworth filter B_m satisfies

$$|\Lambda B_m(f)| = \sqrt{\frac{1}{f^{2m}+1}}.$$

*5.5. Find and plot the impulse response of the normalised lowpass Butterworth filters B_1, B_2 and B_3.

5.6. Plot the signal

$$t\Pi(t) = \begin{cases} t & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

5.7. Let x be the signal with Fourier transform $\hat{x}(f) = \Pi(f)\big(\cos(2\pi f)+1\big)$. Plot the Fourier transform $\hat{x}$ and find and plot x.

5.8. State whether the following signals are bandlimited and, if so, find the bandwidth:

(a) $\operatorname{sinc}(4t)$,
(b) $\Pi(t/4)$,
(c) $\cos(2\pi t)\operatorname{sinc}(t)$,
(d) $e^{-|t|}$.

5.9. Show that

$$\sum_{n\in\mathbb{Z}} e^{\alpha|n|} = 1 + \frac{2}{e^{-\alpha}-1}$$

if $\alpha < 0$ (Hint: solve Exercise 3.7 first).

5.10. Show that if a sequence is absolutely summable then it is also square summable.

5.11. Show that $\sum_{k=0}^{N-1} e^{j2\pi nk/N}$ is equal to N if n is a multiple of N and zero if n is any integer not a multiple of N. (Hint: use the result from Exersise 3.7)

5.12. Let $d = \mathcal{D}_N c$ be the discrete Fourier transform of the sequence c. Show that

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} \qquad n = 0, \dots, N-1.$$

(Hint: use the result from Exersize 5.11)

5.13. Plot the sequence $\cos(n)$ and determine whether it is bounded or periodic.

5.14. Find the discrete time Fourier transform of the sequence $\alpha^n u_n$ where $|\alpha| < 1$ and u_n is the step sequence. Plot the sequence and the magnitude of the discrete time Fourier transform when $\alpha = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}$.

5.15. Given (5.1.3) show that (5.1.4) holds.

$**$5.16. Let x and y be square integrable signals. Show that $\mathcal{F}(xy) = \hat{x} * \hat{y}$.

$**$5.17. Let c be an absolutely summable sequence. Show that

$$\mathcal{F}\sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(t-n) = \sum_{n\in\mathbb{Z}} c_n \mathcal{F}(\operatorname{sinc}(t-n)).$$

$**$5.18. Let c be a square summable sequence and let

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(t-n)$$

be the bandlimited signal with samples $x(n) = c_n$. Show that

$$\mathcal{F}x = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \mathcal{F}\big(\operatorname{sinc}(t-n)\big) \quad \text{a.e.}$$

Find an example where equality does not hold pointwise.

$**$5.19. Let x be an absolutely integrable signal. Show that the periodic summation $\sum_{m\in\mathbb{Z}} x(t+mP)$ is a locally integrable signal. Show that this is not necessarily true if x is square integrable, but not absolutely integrable.

Chapter 6

Discrete-time systems

We have so far studied linear shift-invariant systems and inparticular those systems described by linear differential equations with constant coefficients. Such systems are useful for modelling electrical circuits, mechanical machines, electro-mechanical devices, and many other real world devices. The time-shifter T_τ with non zero time shift $\tau \neq 0$ has so far been absent. We now consider linear shift-invariant systems constructed from linear combinations of time-shifters of the form T_{Pm} where $m \in \mathbb{Z}$ and P is a positive real number called the **sample period** or simply the **period**. That is, we consider systems of the form

$$Hx = \sum_{m \in \mathbb{Z}} h_m T_{Pm} x \tag{6.0.1}$$

where $h \in \mathbb{Z} \to \mathbb{C}$ is a complex valued sequence. Such systems are called **discrete-time systems**.

It is common in the literature that the term discrete-time system refers to a mapping from a complex valued sequence c to another complex valued sequence d of the form

$$d_n = \sum_{m \in \mathbb{Z}} h_m c_{n-m} = (h * c)_n \tag{6.0.2}$$

The sum is called the **discrete convolution** of sequences h and c. The notation $h * c$ is reused to denote discrete convolution so that $d = h * c$. We can recover this definition from (6.0.1) by identifying d_n with $Hx(nP)$ and c_n with $x(nP)$. We will find that (6.0.1) provides a setting for the study of discrete-time systems that is closely linked with previous chapters where what are often called **continuous-time** systems have been studied. Complex exponetial signals e^{st} remain the eigenfunctions and the notions of the transfer function and spectrum carry through unchanged. This definition of discrete-time system (6.0.1) is connected with what Zemanian [1965, Sec. 9.5] calls the continuous variable case of a linear **difference equation** with constant coefficients. We will study difference equations in Section 6.6.

It is often the case that we have access to a sequence of samples $c_n = x(nP)$ at sample rate $F = \frac{1}{P}$ of a signal x of interest. For example, in Tests 1, 2, 3, and 4 we obtained samples of a voltage signal using a computer soundcard. In Tests 7 and 8 we operated on the samples from an audio recording that were stored on a computer harddisk. A convenient property of discrete-time systems is that can operate directly on sampled signals. To see this, let H be a discrete-time system with period P defined as in (6.0.1) and let $y = Hx$ be the response of H to x. It is often the case that we are only interested in the samples of the response $d_n = y(nP) = Hx(nP)$ at sample rate $F = \frac{1}{P}$. For example, in Test 11 we will apply a discrete-time system to a sampled audio recording and the samples of the response will be written to an audio file in `wav` format. If only the samples $d_n = y(nP)$ are required then,

$$\begin{aligned}
d_n &= y(nP) \\
&= Hx(nP) \\
&= \sum_{m \in \mathbb{Z}} h_m T_{mP} x(nP) \\
&= \sum_{m \in \mathbb{Z}} h_m x(nP - mP) \\
&= \sum_{m \in \mathbb{Z}} h_m c_{n-m} = (h * c)_n
\end{aligned} \tag{6.0.3}$$

We see that computing d_n requires only the samples $c_n = x(nP)$ of the input signal x. The values $x(t)$ at times t not a multiple of the sample period are not required. This property makes discrete-time systems particularly convenient for implementation within a computer. Computers are good a storing sequences of numbers and computing sums and products. Observe that by considering only the samples of the response we have recovered the common definition (6.0.2) of a discrete-time system.

Discrete-time systems are not regular because the time-shifter is not regular. However, we will find that the sequence h plays a role analogous to that of the impulse response of a regular system. For this reason h is called the **discrete impulse response** of H.

6.1 The discrete impulse response

We first describe a suitable domain for each discrete-time system. It is worth considering an example. Suppose that H has discrete impulse response given by the step sequence u (5.3.1). The signal $x(t) = 1$ that takes the value 1 for all $t \in \mathbb{R}$ will not be in the domain of H because, in this case,

$$Hx(t) = \sum_{m \in \mathbb{Z}} u_m T_{Pm} x(t) = \sum_{m \in \mathbb{Z}} u_m = \sum_{m=0}^{\infty} 1$$

is not finite for any $t \in \mathbb{R}$. Given a sequence h, denote by $\operatorname{dom}_P h$ the set of signals x such that

$$\sum_{m \in \mathbb{Z}} |h_m x(t - Pm)| < \infty \qquad \text{for all } t \in \mathbb{R}.$$

It can be confirmed that $\operatorname{dom}_P h$ is a linear shift-invariant space. If H has discrete impulse response h then, for all signals $x \in \operatorname{dom}_P h$,

$$|Hx(t)| = \left| \sum_{m \in \mathbb{Z}} h_m T_{Pm} x(t) \right| \leq \sum_{m \in \mathbb{Z}} |h_m x(t - Pm)| < \infty \qquad \text{for all } t \in \mathbb{R},$$

that is, $Hx(t)$ is finite for all $t \in \mathbb{R}$. From here on, we take $\operatorname{dom}_P h$ as the domain of the discrete-time system with discrete impulse response h unless otherwise stated.

The discrete impulse response h immediately yields some properties of the corresponding discrete-time system H. For example, if $h_m = 0$ for all $m < 0$, then H is causal because the response

$$Hx(t) = \sum_{m \in \mathbb{Z}} h_m T_{Pm} x(t) = \sum_{m=0}^{\infty} h_m x(t - Pm)$$

at time t only depends on values of the input signal x at times less than or equal to t. A discrete-time system is stable if and only if its discrete impulse response is absolutely summable (Exercise 6.8). This is analagous to the property of regular systems that are stable if and only if their impulse response is absolutely integrable (Exercise 3.5).

Let F and G be discrete-time systems with equal sample period P and discrete impulse responses f and g. Let $a, b \in \mathbb{C}$ and let $H = aF + bG$ be a system formed by linear combination of F and G. The response of H to input signal $x \in \operatorname{dom}_P f \cap \operatorname{dom}_P g$ is

$$\begin{aligned} Hx &= a \sum_{n \in \mathbb{Z}} f_n T_{Pn} x + b \sum_{n \in \mathbb{Z}} g_n T_{Pn} x \\ &= \sum_{n \in \mathbb{Z}} (a f_n + b g_n) T_{Pn} x, \end{aligned}$$

and so H is a discrete-time system with discrete impulse response given by the linear combination of sequences $af + ag$. We take the domain of H to be $\operatorname{dom}_P f \cap \operatorname{dom}_P g$ unless otherwise stated.

Now suppose that $Hx = FGx$ is formed by the composition of discrete-time systems F and G. Denote by $\operatorname{dom}_P f\, g$ the set of signals x such that

$$\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |f_m g_k x(t-m-k)| < \infty \qquad \text{for all } t \in \mathbb{R}.$$

We will find $\operatorname{dom}_P f\, g$ to be a convenient domain for the system $H = FG$. The response of H to $x \in \operatorname{dom}_P f\, g$ is

$$\begin{aligned} Hx &= \sum_{m\in\mathbb{Z}} f_m T_{Pm} G x \\ &= \sum_{m\in\mathbb{Z}} f_m G T_{Pm} x \qquad \text{(shift-invariance of } G) \\ &= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f_m g_k T_{P(m+k)} x. \end{aligned}$$

Because $x \in \operatorname{dom}_P f\, g$, Fubini's theorem [Rudin, 1986, Theorem 8.8] justifies swapping the order of summation so that

$$Hx = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f_m g_k T_{P(m+k)} x$$

and by putting $n = m + k$ we have

$$Hx = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f_m g_{n-m} T_{Pn} x = \sum_{n\in\mathbb{Z}} h_n T_{Pn} x$$

where h is the sequence with elements

$$h_n = \sum_{m\in\mathbb{Z}} f_m g_{n-m} = (f * g)_n.$$

We have found that the system H constructed by composition of the discrete-time systems F and G is a discrete-time system. The discrete impulse response of H is the discrete convolution of the discrete impulse responses of F and G. The domain of H is taken to be $\operatorname{dom}_P f\, g$ unless otherwise stated.

Discrete convolution has properties analogous to that of the convolution of signals described in Section 3.2. For example, discrete convolution is commutative and associative under approriate assumptions (Exercise 6.11). In what follows we will often use the term convolution and impulse response rather than the lengthier terms discrete convolution and discrete impulse response whenever there is no chance for confusion.

6.2 The transfer function and the spectrum

Recall that complex exponential signals are eigenfunctions of linear shift-invariant systems, that is, if H is a linear shift-invariant system with domain X, then the response of H to input signal $e^{st} \in X$ satisfies

$$He^{st} = \lambda H(s) e^{st}$$

where $\lambda H(s)$ is a complex number that depends on $s \in \mathbb{C}$, but not on $t \in \mathbb{R}$. Considered as a function of s, the expression λH is called the transfer function of the system H. The domain of λH is the set of complex numbers s such that the signal $e^{st} \in X$ and this set is denoted by $\operatorname{cep} X$.

Let H be a discrete-time system with discrete impulse response h and domain $\operatorname{dom}_P h$. The response of H to $e^{st} \in \operatorname{dom}_P h$ satisfies

$$\begin{aligned} He^{st} &= \sum_{n\in\mathbb{Z}} h_n T_{Pn} e^{st} \\ &= \sum_{n\in\mathbb{Z}} h_n e^{s(t-Pn)} \\ &= e^{st} \sum_{n\in\mathbb{Z}} h_n e^{-sPn} = e^{st} \lambda H(s). \end{aligned}$$

It follows that the transfer function of H satisfies

$$\lambda H(s) = \sum_{n\in\mathbb{Z}} h_n e^{-sPn} \qquad s \in \operatorname{cep} \operatorname{dom}_P h.$$

For example, consider the discrete-time system H with discrete impulse response given by the step sequence u. The domain of the transfer function $\operatorname{cep} \operatorname{dom}_P u$ contains all those complex numbers s for which

$$\sum_{n\in\mathbb{Z}} \left| u_n e^{-sPn} \right| = \sum_{n=0}^{\infty} e^{-\operatorname{Re} sPn} < \infty.$$

This is precisely those $s \in \mathbb{C}$ with positive real part. The transfer function is

$$\lambda H(s) = \sum_{n\in\mathbb{Z}} u_n e^{-sPn} = \sum_{n=0}^{\infty} e^{-sPn} = \frac{1}{1 - e^{-sP}} \qquad \operatorname{Re} s > 0.$$

The identity system T_0 is a discrete-time system with discrete impulse response given by the delta sequence

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The transfer function is

$$\lambda T_0(s) = \sum_{n\in\mathbb{Z}} \delta_n e^{-sPn} = 1 \qquad s \in \mathbb{C}.$$

The domain of λT_0 is the entire complex plane $\mathbb{C}$. This agrees with (4.3.1).

Now suppose that H is a stable discrete-time system with absolutely summable discrete impulse response h. The oscillatory complex exponential signal $e^{2\pi jft}$ is in the domain $\operatorname{dom}_P h$ for all $f \in \mathbb{R}$ because

$$\sum_{m\in\mathbb{Z}} \left| h_m e^{2\pi jf(t-m)} \right| = \sum_{m\in\mathbb{Z}} |h_m| < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Equivalently, the domain of the transfer function λH contains the imaginary axis, that is, the set $\operatorname{cep}\operatorname{dom}_P h$ contains the imaginary axis. The spectrum of H is

$$\Lambda H(f) = \lambda H(j2\pi f) = \sum_{n\in\mathbb{Z}} h_n e^{-2\pi jfPn} \qquad f \in \mathbb{R}.$$

Observe that the spectrum ΛH is a periodic signal with period equal to the reciprocal of the sample period $F = \frac{1}{P}$ called the **sample rate**.

The spectrum is related to the discrete-time Fourier transform (5.5.3) of h by

$$\mathcal{D}h(f) = \Lambda H(\tfrac{f}{P}) = \sum_{n\in\mathbb{Z}} h_n e^{-2\pi jfn}.$$

As an example, suppose that H is the system with period P and discrete impulse response

$$h_n = \Pi(n/2) = \begin{cases} 1 & |n| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The spectrum of H is

$$\Lambda H(f) = \mathcal{D}h(Pf) = e^{-2\pi jPf} + 1 + e^{2\pi jPf} = 1 + 2\cos(2\pi Pf).$$

The sequence $\Pi(n/2)$ and its discrete-time Fourier transform are plotted in Figure 5.10.

6.3 Ideal digital filters

We have found that the spectrum of a stable discrete-time system is periodic with period equal to the sample rate $F = \frac{1}{P}$. Because of this periodicity it is not immediately apparent that any useful filters can be constructed from discrete-time systems. For example, periodicity makes it impossible to build a lowpass filter (Section 5.2). However, many signals occurring in practice are approximately bandlimited with some bandwidth $b > 0$ (Section 5.4). We will find that it is possible to perform useful filtering operations on bandlimited signals using discrete-time systems. This is most easily achieved when the bandwidth is less than half the sample rate of the system, that is, when $b < \frac{F}{2}$.

To see how this works, let x be the bandlimited signal with Fourier transform

$$\hat{x}(f) = \mathcal{F}x(f) = \frac{4}{3}\Pi(f) - \cos(2\pi f)\big(\Pi(2f-3) + \Pi(2f+3)\big)$$

One can show that the signal x takes the form (Exercise 6.1)

$$x(t) = \frac{4}{3}\operatorname{sinc}(f) + \frac{1}{2}\cos(3\pi t)\left(\operatorname{sinc}\left(\frac{t+1}{2}\right) + \operatorname{sinc}\left(\frac{t-1}{2}\right)\right) \qquad (6.3.1)$$

The Fourier transform $\hat{x}$ is plotted by the dashed line in Figures 6.1 and 6.4. The Fourier transform has a rectangular pulse of height $\frac{4}{3}$ centered at the origin and cosine pulses of width $\frac{1}{2}$ at $f = \pm\frac{3}{2}$. The signal is bandlimited with bandwidth $\frac{7}{4}$. Suppose that the component of interest is the rectangular pulse at the origin and that the two cosine pulses would preferably not exist. A low pass filter could be used to filter out the cosine pulses. For example, an ideal lowpass filter L_c (Section 5.2) with cuttoff frequency c in the interval $(\frac{1}{2}, \frac{5}{4})$ and spectrum $\Pi(\frac{f}{2c})$ would preserve the rectangular pulse at the origin and remove the two cosine pulses. We are lead to believe that the response $L_c x$ of the filter L_c to input signal x has Fourier transform (5.0.4)

$$\mathcal{F}L_c x = \Lambda L_c \mathcal{F}x = \Pi(\tfrac{f}{2c})\hat{x}(f) = \tfrac{3}{2}\Pi(f).$$

The impulse response of this ideal lowpass filter L_c is $2c\sin(2ct)$.

Because the signal x has bandwidth $b = \frac{7}{4}$ the spectrum of the filter for frequencies $|f| > \frac{7}{4}$ is not important. By exploiting this property we will find that the same lowpass filtering effect can be achieved with a discrete-time system. Denote by L_c^P the discrete-time system with sample period $P = \frac{1}{F}$ and discrete impulse response $h_n = 2cP\operatorname{sinc}(2cPn)$. That is, h is given by multiplying the impulse response of the ideal low pass filter L_c by P and then sampling with period P. From (5.5.5), the discrete-time Fourier transform of h is given by periodic summation of $\Pi(Ff/2c)$, that is,

$$\hat{h}(f) = \mathcal{D}h(f) = \sum_{m\in\mathbb{Z}} \Pi\left(\frac{Ff + Fm}{2c}\right).$$

We are lead to believe that the spectrum of L_c^P is

$$\Lambda L_c^P(f) = \hat{h}(Pf) = \sum_{m\in\mathbb{Z}} \Pi\left(\frac{f + Fm}{2c}\right). \qquad (6.3.2)$$

The spectrum is periodic with period F and is plotted by the shaded region in Figure 6.1 in the case that $c = 1$ and $F = \frac{1}{P} = 4$. When $F > 2c$, the system L_c^P is called an **ideal lowpass digital filter** with cuttoff frequency c and sample period P. Now consider the response of L_c^P to the signal x

from (6.3.1). If $c \in (\frac{1}{2}, \frac{5}{4})$ and $F > \frac{7}{2}$ then the response $L_c^P x$ has Fourier transform

$$\mathcal{F} L_c^P x = \Lambda L_c^P \, \mathcal{F} x = \hat{x}(f) \sum_{m \in \mathbb{Z}} \Pi\left(\frac{f + Fm}{2c}\right) = \tfrac{4}{3}\Pi(f)$$

exactly as with the ideal lowpass filter L_c. This is seen in Figure 6.1 (top) where only the period at the origin of the spectrum $\Lambda L_1^{1/4}$ overlaps with the Fourier transform of the bandlimited signal x.

The conclusion that the spectrum of the ideal lowpass digital filter L_c^P satisfies (6.3.2) needs to be taken with some scepticism because the discrete impulse response $h_n = 2cP \operatorname{sinc}(2cPn)$ is not absolutely summable and so the domain $\operatorname{cep\,dom}_P h$ of ΛL_c^P does not contain the imaginary axis. Like ideal analogue filters (Section 5.2), ideal digital filters will not be useful in practice and so this caveat will not affect the results that follow. The concept of ideal filters is nonetheless a valuable aid to intuition.

Ideal highpass and ideal bandpass digital filters can be constructed using L_c^P in much the same way that ideal analogue filters were constructed using the ideal lowpass filter L_c in Section 5.2. An **ideal high pass digital filter** with cuttoff frequency c and period $P = \frac{1}{F} < \frac{1}{2c}$ is the system given by the linear combination

$$T_0 - L_c^P.$$

The impulse response is $\delta_n - 2cP \operatorname{sinc}(2cnP)$. The spectrum

$$\Lambda(T_0 - L_c^P) = 1 - \Lambda L_c^P = 1 - \sum_{m \in \mathbb{Z}} \Pi\left(\frac{f + Fm}{2c}\right)$$

is plotted in Figure 6.1 (middle) in the case that $c = 1$ and $F = \frac{1}{P} = 4$. The Fourier transform of the response of $T_0 - L_1^{1/4}$ to input signal x from 6.3.1 is plotted in Figure 6.1 (middle). The two higher frequency cosine pulses are preserved and the rectangular pulse at the origin is removed.

An **ideal bandpass digital filter** with upper cuttoff frequency u, lower cuttoff frequency ℓ, and period P is the system given by the linear combination

$$L_u^P - L_\ell^P.$$

The impulse response is the sequence $2uP \operatorname{sinc}(2unP) - 2\ell P \operatorname{sinc}(2\ell nP)$. The spectrum $\Lambda L_u^P - \Lambda L_\ell^P$ is plotted in Figure 6.1 (bottom) in the case that $u = \frac{3}{2}$, $\ell = \frac{1}{4}$, and $F = \frac{1}{P} = 4$. The Fourier transform of the response of $L_u^P - L_\ell^P$ to input x from (6.3.1) is also plotted.

6.4 Finite impulse response filters

Ideal digital filters are not realisable in practice. We now turn our attention to practical digital filters. The first of these are called **finite impulse**

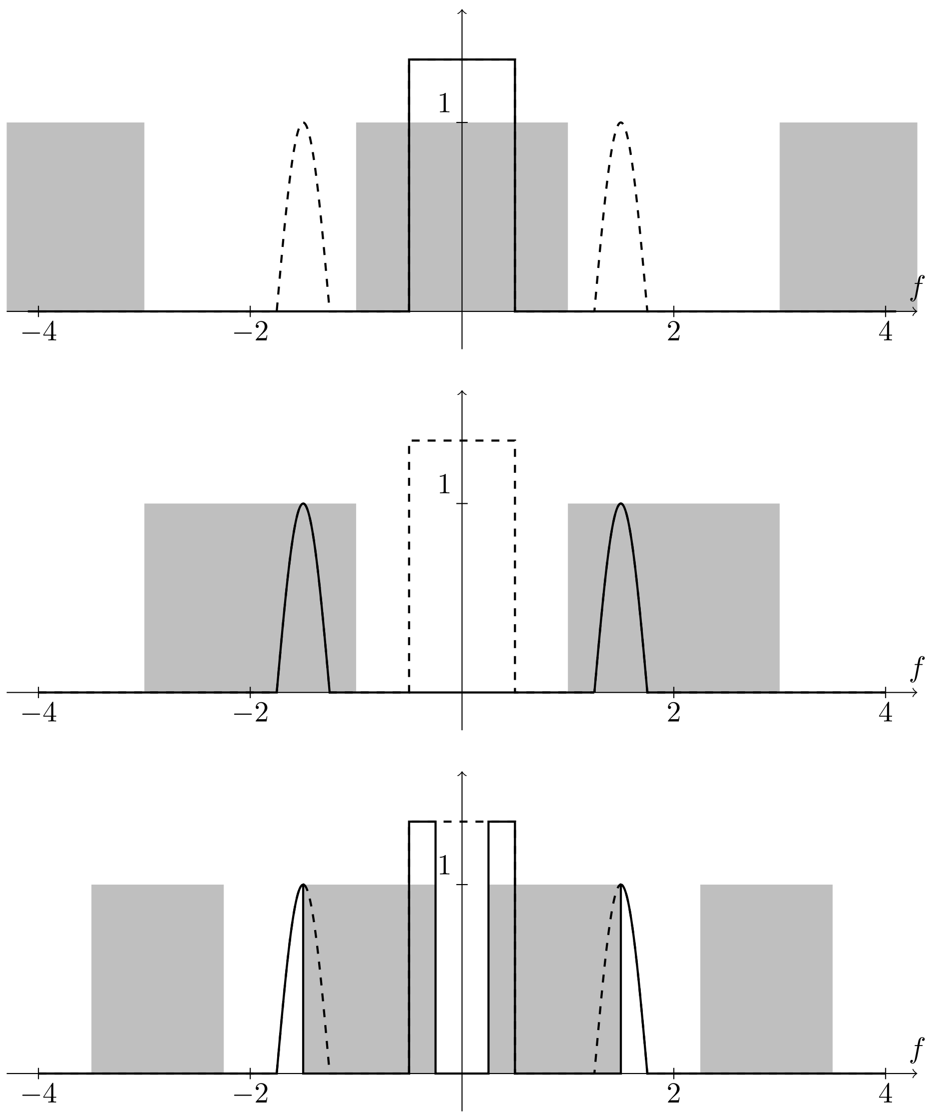

Figure 6.1: Fourier transform $\frac{4}{3}\Pi(f)-\cos(2\pi f)\big(\Pi(2f-3)+\Pi(2f+3)\big)$ of a bandlimited signal with bandwidth $b \leq \frac{7}{4}$ (dashed). The solid lines shows the magnitude of the Fourier transform of the response to this signal to the ideal lowpass digital filter $L_1^{1/4}$ (top), the ideal highpass digital filter $T_0 - L_1^{1/4}$ (middle), and the ideal bandpass digital filter $L_{3/2}^{1/4} - L_{1/4}^{1/4}$. The spectra of these filters are shown by the shaded regions. The spectra have period $P = 4$. The periodicity is not of consequence because the signal is bandlimited with bandwidth $b \leq \frac{7}{4} < \frac{F}{2} = 2$.

response filters or **FIR filters** for short. One feature preventing practical implementation of the ideal lowpass digital filter L_c^P is that the discrete impulse response $h_n = 2cP\sin(2cPn)$ is not of finite support. Computing the response $\sum_{m\in\mathbb{Z}} h_m T_{Pm} x(t)$ for any time $t \in \mathbb{R}$ might require summing an infinite number of terms. A pragmatic approach to alleviate this problem is to first approximate the ideal analogue lowpass filter L_c by a regular system with impulse response

$$2cw(t)\,\mathrm{sinc}(2ct)$$

where $w(t)$ is a bounded signal called a **window function** or **apodization function**. The window function $w(t)$ is zero outside an interval $(-W/2, W/2)$ where $W > 0$ is called the **window width**. Because of the window function this impulse response has finite support and is absolutely integrable.

Common window functions include the rectangular window $w(t) = \Pi(t/W)$, the triangular (or **Bartlett**) window $w(t) = \triangle(2t/W)$ where

$$\triangle(t) = (\Pi * \Pi)(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(Exercise 5.2)}$$

the raised cosine (or **Hann**) window

$$w(t) = \frac{1}{2}\Pi\left(\frac{t}{W}\right)\left(1 + \cos\left(\frac{2\pi t}{W}\right)\right),$$

and the **Blackman window** [Blackman and Tukey, 1959]

$$w(t) = \Pi\left(\frac{t}{W}\right)\left(a_0 + a_1\cos\left(\frac{2\pi t}{W}\right) + a_2\cos\left(\frac{4\pi t}{W}\right)\right) \qquad (6.4.1)$$

where $a_0 = \frac{21}{50}$, $a_1 = \frac{1}{2}$, and $a_2 = \frac{2}{25}$. These window functions and their Fourier transforms are plotted in Figure 6.2. The Fourier transform of the Blackman window decays to zero particularly rapidly as $|f|$ increases (Exercise 6.2).

From the multiplication property of the Fourier transform (5.1.6) we find the Fourier transform of the windowed sinc function $2cw(t)\,\mathrm{sinc}(2ct)$ to be

$$\mathcal{F}\big(2cw(t)\,\mathrm{sinc}(2ct)\big) = \hat{w}(f) * \Pi(f/2c),$$

that is, the convolution of the Fourier transform of the window function $\mathcal{F}w = \hat{w}$ and the Fourier transform $\mathcal{F}\big(2c\,\mathrm{sinc}(2ct)\big) = \Pi(f/2c)$. This Fourier transform is plotted in Figure 6.3 for the rectangular and Blackman windows when the cuttoff $c = 1$ and the width $W = 1, 3, 10$. The Fourier transform approaches the spectrum of the ideal lowpass filter as the window width W increases.

A **low pass finite impulse response filter** with period P, cuttoff frequency c, and window w is a discrete-time system $L_c^{P,w}$ with period P having impulse response $h_n = 2cPw(nP)\operatorname{sinc}(2cnP)$. That is, h is given by multiplying the windowed sinc function by P and sampling with period P. Because $w(t)$ is zero outside the interval $(-W/2, W/2)$, it follows that $h_n = 0$ when $nP \notin (-FW/2, FW/2)$ and so only a finite number of elements of h are nonzero. For this reason, the response of $L_c^{P,w}$ to any input signal x can be computed by the finite sum

$$L_c^{P,w}x(t) = \sum_{m=-a}^{a} h_m T_{Pm} x(t) \qquad \text{where } a = \lfloor FW/2 \rfloor \tag{6.4.2}$$

and $\lfloor FW/2 \rfloor$ denotes the greatest integer less than or equal to $FW/2$. The number of terms to be summed is $2a+1$. This is typically called the number of filter **taps**. The number of non zero terms in the impulse response is at most the number of taps.

The spectrum of this filter is given by periodic summation of the Fourier transform $\hat{w}(f) * \Pi(f/2c)$ of the windowed sinc function, that is,

$$\Lambda L_c^{P,w}(f) = \sum_{m\in\mathbb{Z}} \big(\hat{w}(t) * \Pi(f/2c)\big)(Ff + Fm) = \mathcal{D}h(Pf).$$

The magnitude spectrum $|\Lambda L_c^{P,w}(f)|$ is shown by the shaded regions in Figure 6.4 for the rectangular and Blackman windows.

Now consider filtering the signal x from (6.3.1) using a low pass finite impulse response filter $L_c^{P,w}$. Figure 6.4 shows the response in the case that the cuttoff frequency $c = 1$, the sample rate $F = \frac{1}{P} = 4$, and for the rectangular window with window width $W = 3$ and 8 and a Blackman window with width $W = 8$. The filter using the Blackman window is particularly effective. Its response is close to that of the ideal lowpass filter L_1 from Figure 6.1. The rectangular window suffers from ocillations that decay slowly as the window width W increases.

In the following Test 11 we design a low pass digital filter for the audio taken from lecture recording used in Tests 7, 8, and 10.

Test 11 (**Finite impulse response filtered lecture recording**)

We consider again the 20 s audio signal from Test 7. The mono audio recording contains $N = 440998$ samples denoted by $c_0, c_1, \dots, c_{N-1}$ at sample rate $F = 22\,050\,\text{Hz}$. We put $c_n = 0$ when $n < 0$ or $n \geq N$. Let x be the bandlimited signal with samples $c_n = x(nP)$, that is,

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(Ft - n).$$

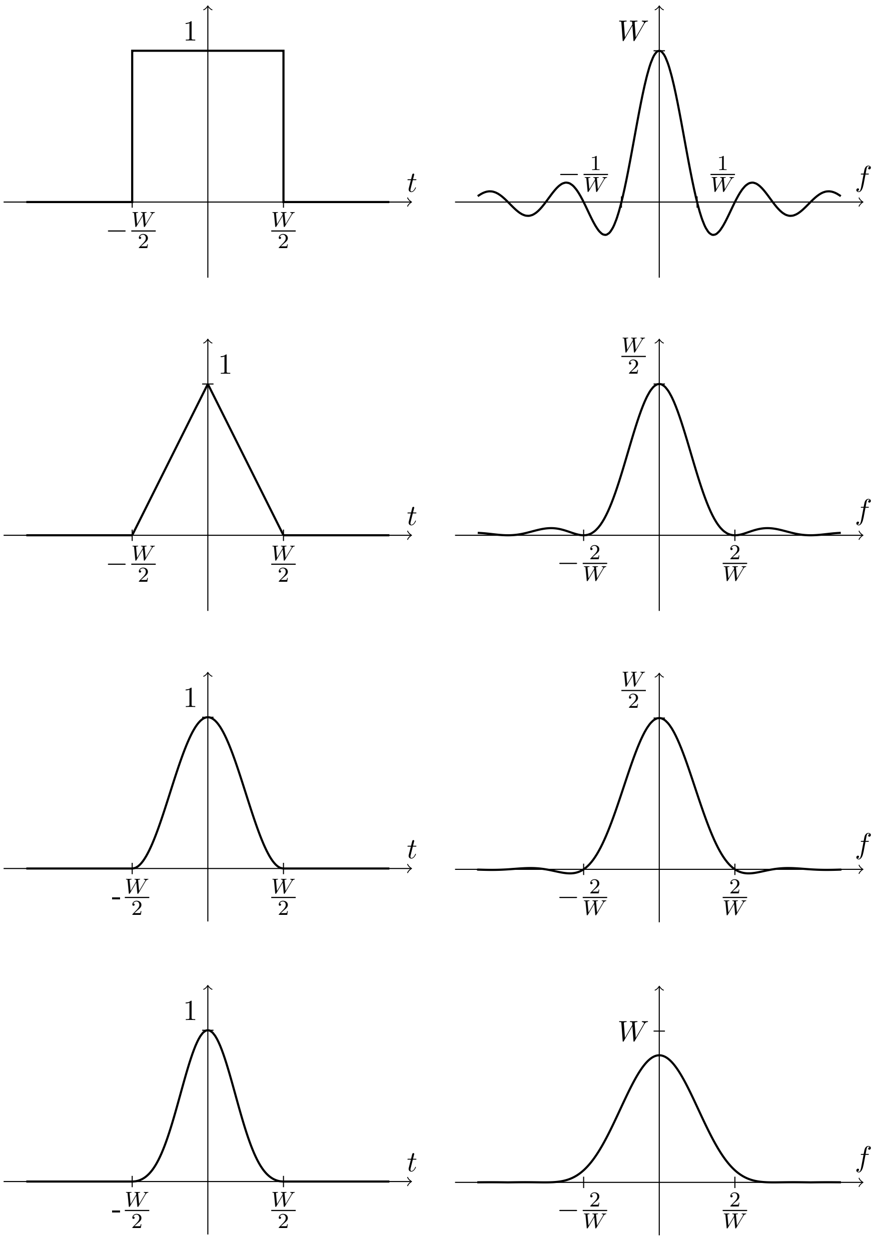

Figure 6.2: Rectangular (top), triangular, Hann, and Blackman (bottom) window functions and their Fourier transforms. The Fourier transforms have zero imaginary part and so only the real part is plotted. The Fourier transform of the Blackman window decays to zero particularly rapidly as $|f|$ increases.

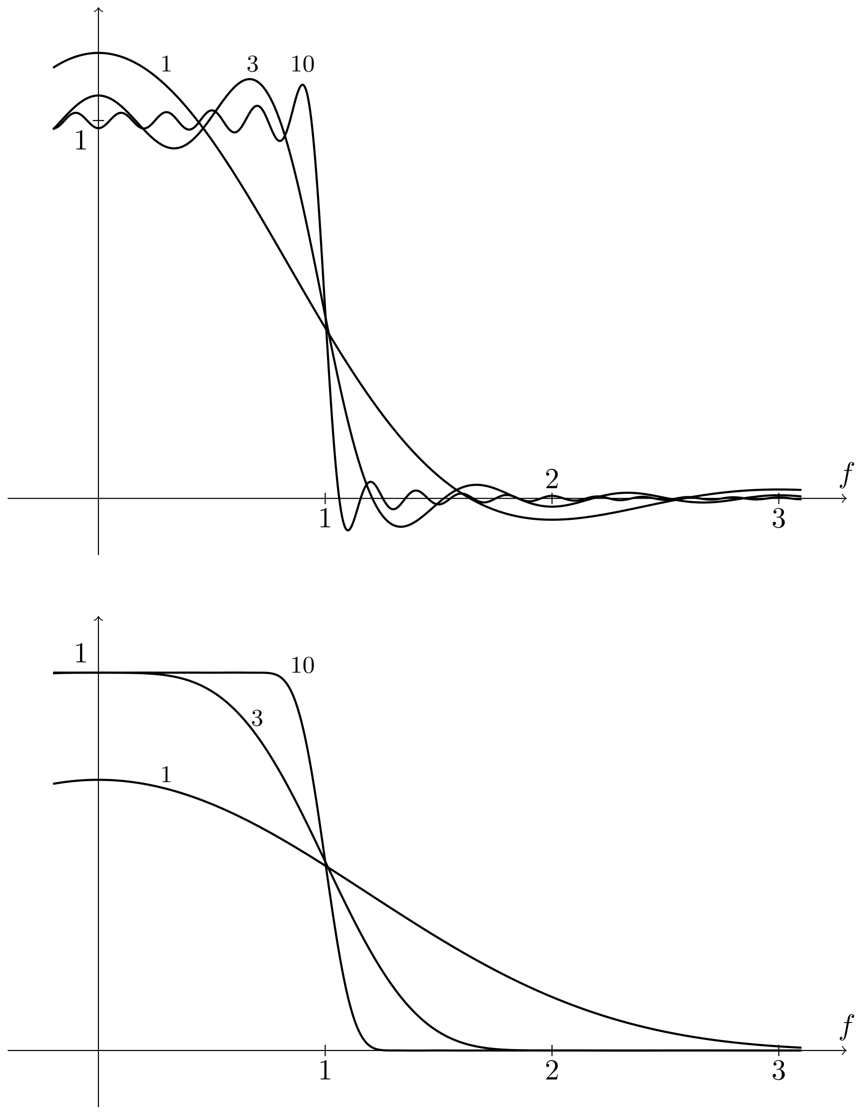

Figure 6.3: Fourier transform $\hat{w}(f) * \Pi(f)$ of windowed sinc functions for window width $W = 1, 3$, and 10. Top: rectangular window $w(t) = \Pi(t/W)$. Bottom: Blackman window (6.4.1). The Fourier transforms approach $\Pi(f)$ as W increases. The rectangular window exhibits large oscillations even when W is large.

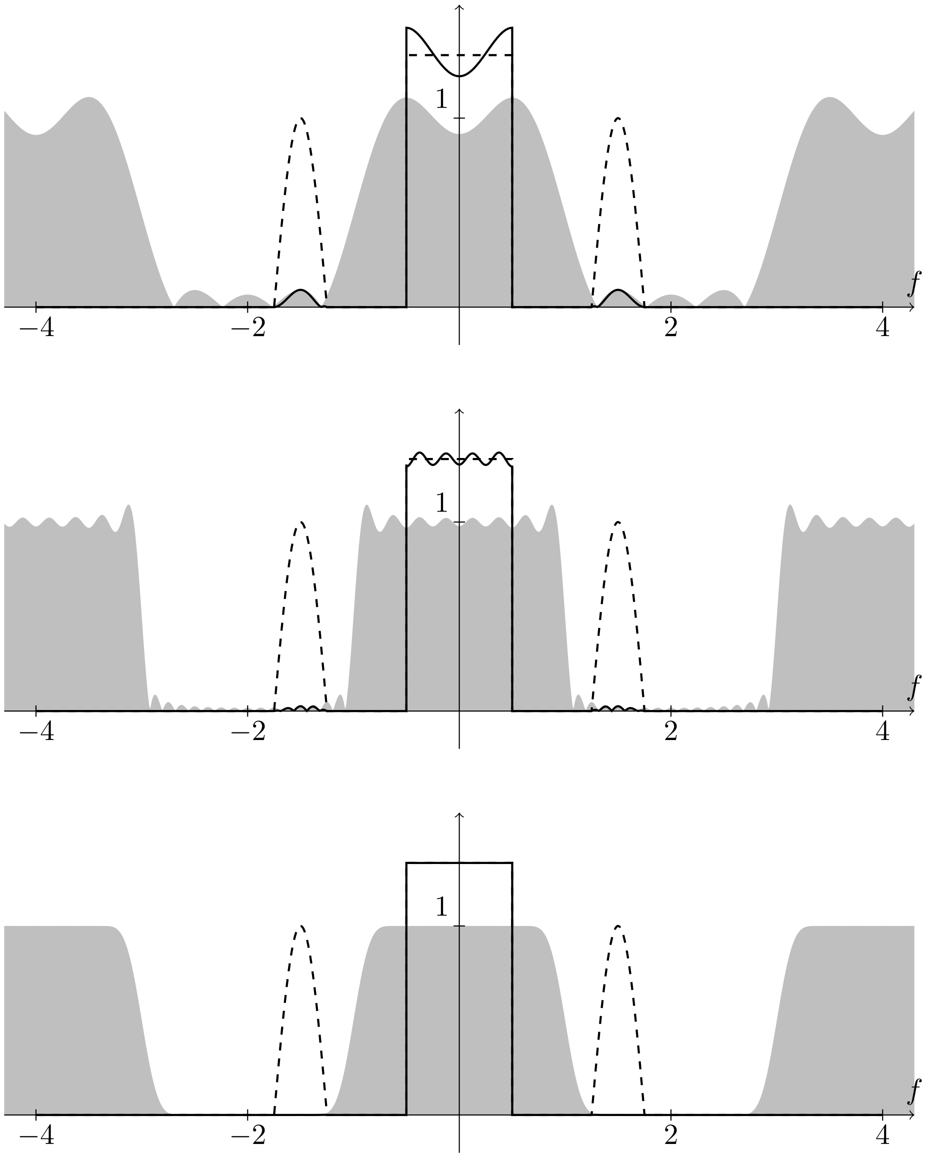

Figure 6.4: Fourier transform $\frac{4}{3}\Pi(f) - \cos(2\pi f)\big(\Pi(2f-3) + \Pi(2f+3)\big)$ of a bandlimited signal with bandwidth $b \leq \frac{7}{4}$ (dashed). The solid lines shows the magnitude of the Fourier transform of the response of the low pass digital finite impulse response filter $L_c^{P,w}$ with cuttoff frequency $c = 1$, period $P = \frac{1}{4}$, and with a rectangular window of width $W = 3$ (top) and width $W = 8$ (middle), and with a Blackman window of width $W = 8$ (bottom). The magnitude spectrum of these filters is shown by the shaded region. The filter using the Blackman window is particularly effective. Its response is close to that of the ideal filter L_1 from Figure 6.1. The rectangular window suffers from ocillations that decay slowly as the window width W increases.

In Test 7 the Fourier transform of x was computed and plotted in Figure 5.8. The audio recording contains human voice primarily residing at lower frequencies below 4 kHz. Audible in the recording is a faint high pitched hum. This hum is represented in Figure 5.8 by spikes occurring at approximately ± 8 kHz and also by the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated. In this test, a low pass digital finite impulse response filter $L_\gamma^{P,w}$ will be used to remove this hum.

We choose the period of the filter to be $P = \frac{1}{F}$ where $F = 22\,050$ Hz is the sample rate of the audio recording. The cuttoff frequency is chosen to be $\gamma = 4600$ Hz and a Blackman window (6.4.1) of width $W = \frac{12}{\gamma} = \frac{1}{400}$ is used. This filter has impulse response

$$h_n = 2\gamma P w(nP)\,\mathrm{sinc}(2\gamma P n)$$

where $w(t)$ is the Blackman window (6.4.1). The impulse response satisfies $h_n = 0$ for those n outside the interval $[-a, a]$ where

$$a = \left\lfloor \frac{FW}{2} \right\rfloor = \left\lfloor \frac{735}{32} \right\rfloor = 28.$$

This filter has $2a + 1 = 57$ taps and can be implemented by a finite sum of 57 terms (6.4.2). The spectrum of the filter is

$$\Lambda L_\gamma^{P,w}(f) = \mathcal{D}h(Pf) = \sum_{n=-a}^{a} h_n e^{-2\pi j P f n}.$$

The magnitude spectrum $|\Lambda L_\gamma^{P,w}|$ is plotted in Figure 6.5.

Let $y = L_\gamma^{P,w} x$ be the response of the filter to the bandlimited audio signal x. We will only have need of the samples of the response $d_n = y(nP) = L_\gamma^{P,w} x(nP)$. From (6.0.3) the sequence d of samples is the discrete convolution of the impulse response h and the sequence c of audio samples, that is,

$$d_n = (h * c)_n = \sum_{m \in \mathbb{Z}} h_m c_{n-m} = \sum_{m=-a}^{a} h_m c_{n-m}.$$

Observe that computing d_n for any n involves only a finite sum of $2a+1 = 57$ terms because $h_n = 0$ for $n > a$ or $n < -a$. The samples $d_0, d_1, \dots, d_{N-1}$ are written to the file `nohum.wav`. Listening to this file confirms that the high pitched hum is no longer audible.

The response y is the bandlimited signal with samples $d_n = L_\gamma^{P,w}$, that is,

$$y(t) = \sum_{n \in \mathbb{Z}} d_n \,\mathrm{sinc}(Ft - n).$$

The Fourier transform of y is given by $\mathcal{F}y(f) = P\Pi(Pf)\mathcal{D}d(Pf)$ where $\mathcal{D}d$ is the discrete Fourier transform of the sequence d (5.5.3). The magnitude of the Fourier transform is plotted in Figure 6.6. The spikes at ± 8 kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist.

6.5 The z-transform

Recall that the transfer function λH of a discrete-time system H with impulse response h satisfies

$$\lambda H(s) = \sum_{n\in\mathbb{Z}} h_n e^{-sPn} \qquad s \in \operatorname{cep}\operatorname{dom}_P h.$$

Putting $z = e^{Ps}$ we have

$$\lambda H(s) = \lambda H(\tfrac{1}{P}\log z) = \sum_{n\in\mathbb{Z}} h_n z^{-n} \tag{6.5.1}$$

that holds for complex numbers $z = e^{sP}$ such that $s \in \operatorname{cep}\operatorname{dom}_P h$. Denote this set by $\operatorname{roc}_z h$, that is,

$$\operatorname{roc}_z h = \{z \in \mathbb{C} \,;\, \frac{1}{P}\log z \in \operatorname{cep}\operatorname{dom}_P h\ \}.$$

The right hand side of (6.5.1) is called the **z-transform** of the sequence h and the set $\operatorname{roc}_z h$ is called the region of convergence of the z-transform.

Let c be a complex valued sequence. The z-transform of c is denoted by $\mathcal{Z}c$. It is the complex valued function of $\operatorname{roc}_z c$ satisfying

$$\mathcal{Z}c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n} \qquad z \in \operatorname{roc}_z c.$$

The domain of $\mathcal{Z}c$ is the region of convergence $\operatorname{roc}_z c$. It happens that the region of convergence $\operatorname{roc}_z c$ is precisely the set of nonzero complex numbers such that the sequence $c_n z^{-n}$ is absolutely summable, that is, those $z \neq 0$ such that

$$\sum_{n\in\mathbb{Z}} \left|c_n z^{-n}\right| < \infty. \qquad \text{(Exercise 6.10)}$$

The z-transform of a complex valued sequence plays a role analogous to the Laplace transform of a signal. Recall that the region of convergence of the Laplace transform was either a half plane, a vertical strip, the entire complex plane, or the empty set (Section 4). We will find that the region of convergence of the z-transform is either a circular disk with the origin

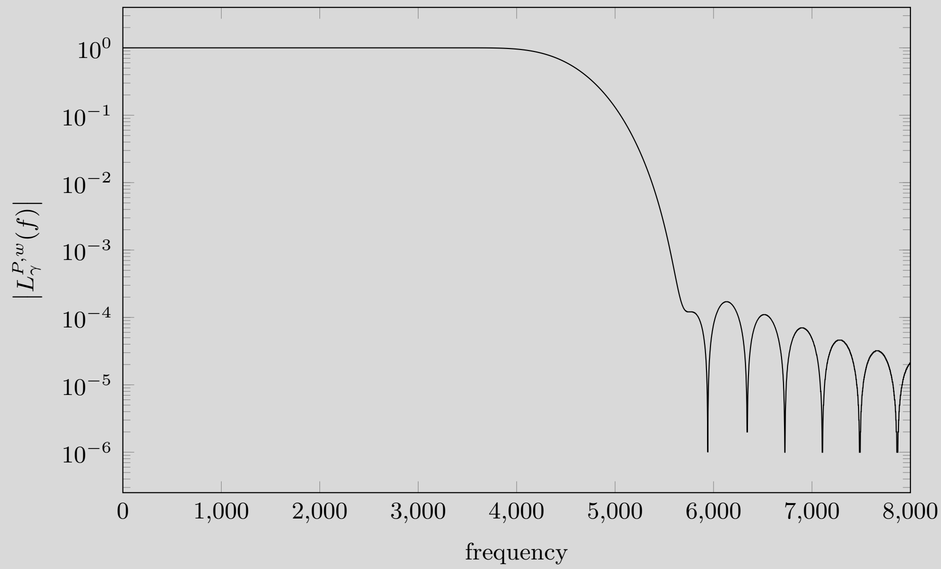

Figure 6.5: Magnitude spectrum of the low pass finite impulse response digital filter $L_\gamma^{P,w}$ with cuttoff frequency $\gamma = 4600\,\text{Hz}$, sample rate $F = \frac{1}{P} = 22\,050\,\text{Hz}$, and Blackman window of width $W = \frac{12}{\gamma} = \frac{1}{400}$. The vertical axis uses a logarithmic scale.

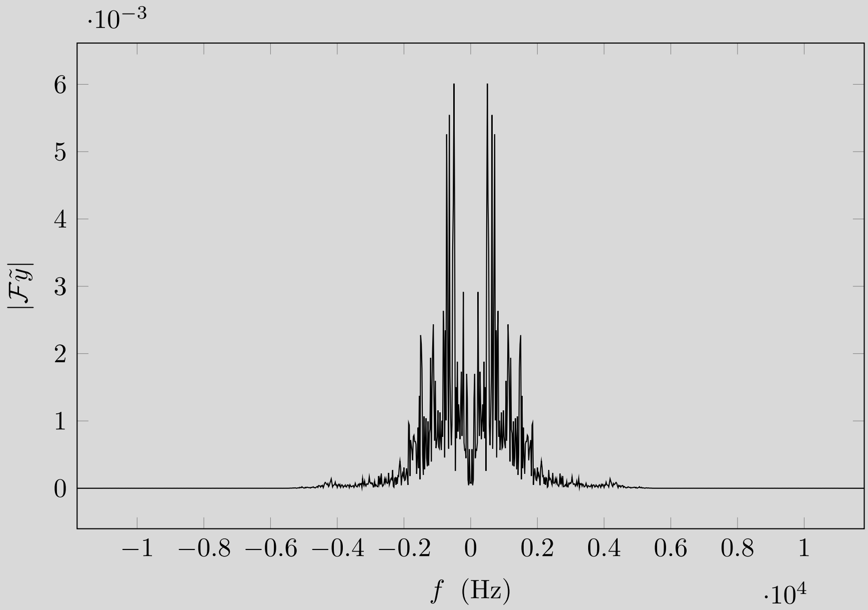

Figure 6.6: Plot of the magnitude of the Fourier transform $\mathcal{F}y$ of the filtered audio signal $y = L_\gamma^{P,w}x$. The plot looks similar to Figure 5.8 except that the spikes at $\pm 8\,\text{kHz}$ and the elevated region between 4900 Hz and 5900 Hz no longer exist.

removed, an annular region in the complex plane, the complex plane with an origin centered disc removed, the entire complex plane, or the empty set.

We now consider some example z-transforms. Consider first the step sequence u. The region of convergence $\mathrm{roc}_z\, u$ is all those complex numbers with magnitude greater than one because

$$\sum_{n\in\mathbb{Z}} |u_n z^{-n}| = \sum_{n=0}^{\infty} |z|^{-n} < \infty \qquad \text{only if } |z| > 1.$$

Graphically this region of convergence is the complex plane with a disc of radius one centered at the origin removed (Figure 6.7). The z-transform of u is

$$\mathcal{Z}u(z) = \sum_{n\in\mathbb{Z}} u_n z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1} \qquad |z| > 1. \qquad \text{(Exercise 6.3)}$$

Now consider the sequence with nth element $(\frac{1}{2})^n u_n$. The region of convergence is all those complex numbers with magnitude greater than $\frac{1}{2}$. The z-transform is

$$\mathcal{Z}\big((\tfrac{1}{2})^n u_n\big) = \sum_{n\in\mathbb{Z}} (\tfrac{1}{2})^n u_n z^{-n} = \sum_{n=0}^{\infty} (2z)^{-n} = \frac{2z}{2z-1} \qquad |z| > \tfrac{1}{2}.$$

Graphically, the region of convergence is the complex plane with a disc of radius $\frac{1}{2}$ removed. Now consider the sequence with elements

$$(\tfrac{3}{2})^n u_{-n} = \begin{cases} (\tfrac{3}{2})^n & n \le 0 \\ 0 & n > 0. \end{cases}$$

In this case the region of convergence is all those nonzero complex numbers with magnitude less than $\frac{3}{2}$ because

$$\sum_{n\in\mathbb{Z}} \big|(\tfrac{3}{2})^n u_{-n} z^{-n}\big| = \sum_{n=0}^{\infty} \big|\tfrac{2}{3} z\big|^n < \infty \qquad \text{only if } |z| < \tfrac{3}{2}.$$

The z-transform is

$$\mathcal{Z}\big((\tfrac{3}{2})^n u_{-n}\big) = \sum_{n\in\mathbb{Z}} (\tfrac{3}{2})^n u_{-n} z^{-n} = \sum_{n=0}^{\infty} (\tfrac{2}{3} z)^n = \frac{3}{3-2z} \qquad |z| < \tfrac{3}{2}$$

The region of convergence is an open disc of radius $\frac{3}{2}$ with the origin removed. The sequence with nth element $(\frac{1}{2})^n u_n + (\frac{3}{2})^n u_{-n}$ has z-transform

$$\mathcal{Z}\big((\tfrac{1}{2})^n u_n + (\tfrac{3}{2})^n u_{-n}\big) = \frac{2z}{2z-1} + \frac{3}{3-2z} \qquad \tfrac{1}{2} < |z| < \tfrac{3}{2}$$

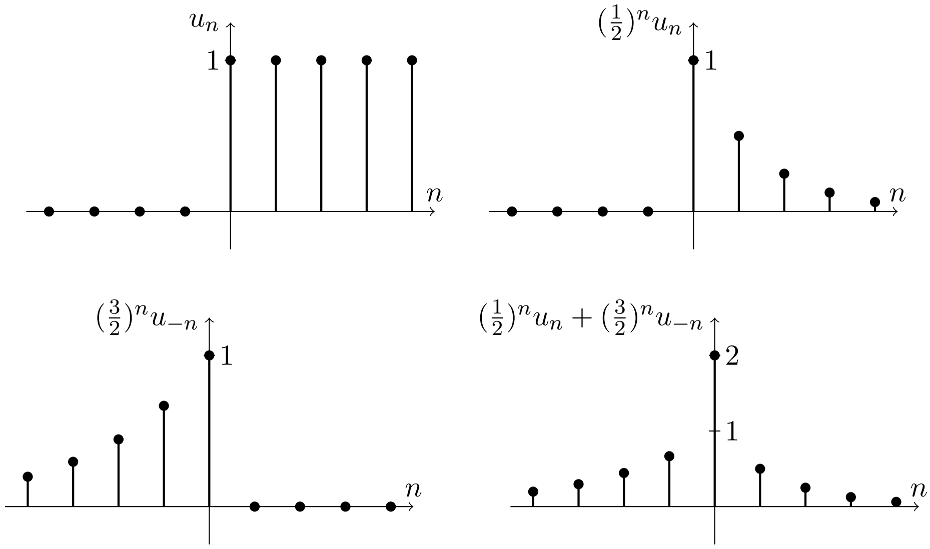

Figure 6.6: Real valued sequences. The top left plot shows the step sequence u.

with region of convergence all those complex numbers such that $\frac{1}{2} < |z| < \frac{3}{2}$. The region of convergence is an annulus (a doughnut) in the complex plane with inner radius $\frac{1}{2}$ and outer radius $\frac{3}{2}$.

The delta sequence δ has z-transform

$$\mathcal{Z}(\delta) = \sum_{n \in \mathbb{Z}} \delta_n z^{-n} = 1.$$

The region of convergence is the entire complex plane with the origin removed, that is, $\text{roc}_z\, \delta = \mathbb{C} \backslash 0$. Finally, consider the sequence $\mathbf{1}$ that has every element equal to 1. Because $\sum_{n \in \mathbb{Z}} |z|^{-n}$ does not converge for any $z \in \mathbb{C}$ the region of convergence $\text{roc}_z\, \mathbf{1}$ is the empty set. The sequence $\mathbf{1}$ is said not to have a z-transform.

Given the z-transform $\mathcal{Z}c$ the sequence c can be recovered by the inverse z-transform

$$c_n = \frac{1}{2\pi j} \oint_C \mathcal{Z}c(z) z^{n-1} dz$$

where C is a counterclockwise closed path encircling the origin and within the region of convergence $\text{roc}_z\, c$. Similarly to the inverse Laplace transform (Section 4), direct calculation of the inverse z-transform requires a form of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following z-transform pair

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1 \qquad \text{(Exercise 6.4)}$$

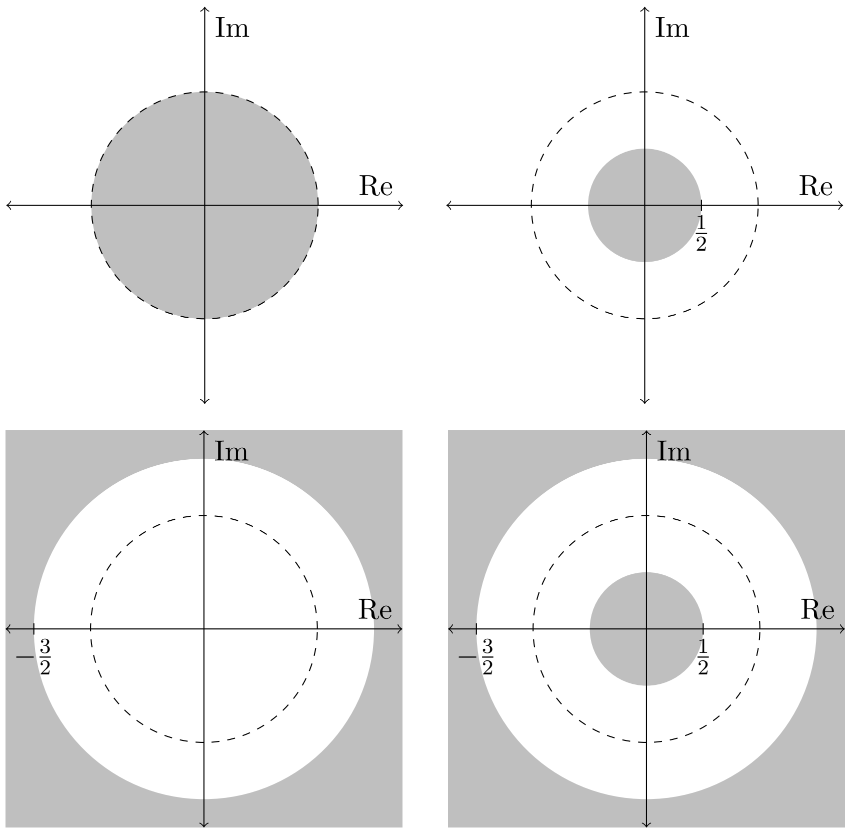

Figure 6.7: Regions of convergence of the z-transforms of (unshaded) the step sequence u (top left), the sequence $(\frac{1}{2})^n u_n$ (top right), the sequence $(\frac{3}{2})^n u_{-n}$ (bottom left), and the sequence $(\frac{1}{2})^n u_n + (\frac{3}{2})^n u_{-n}$ (bottom right). The unit circle is indicated by the dashed circle. The region of convergence takes the form of the complex plane with a disc an origin centered disc removed (top), a disc with the origin removed (bottom left), an annulus (bottom right), the entire complex plane, or the empty set.

where

$$[n]_k = n(n-1)\dots(n-k+1)$$

is called the **falling factorial** [Graham et al., 1994, p. 48]. In the case that $k = 0$ the falling factorial is defined as $[n]_0 = 1$ for all $n \in \mathbb{Z}$.

Let $a \in \mathbb{C}$. The z-transforms of the sequence c_n and the sequence with nth element $a^n c_n$ are related by

$$\mathcal{Z}(a^n c_n)(z) = \sum_{n\in\mathbb{Z}} a^n c_n z^{-n} = \sum_{n\in\mathbb{Z}} c_n (z/a)^{-n} = \mathcal{Z}c(z/a) \qquad \frac{z}{a} \in \operatorname{roc}_z c.$$

This is called the **scaling** property of the z-transform. The region of convergence of $\mathcal{Z}(a^n c_n)$ is the set of complex numbers z such that $z/a \in \operatorname{roc}_z c$. Using this property the z-transform of the sequence $a^n [n]_k u_n$ is

$$\mathcal{Z}\big(a^n [n]_k u_n\big) = \frac{k! a^k z}{(z-a)^{k+1}} \qquad |z| > |a|\,. \tag{6.5.2}$$

This is the only z-transform pair we require here. We will have particular use of the case when k is 0 or 1. In the case that $k = 0$ we obtain the z-transform pair

$$\mathcal{Z}\big(a^n u_n\big) = \frac{z}{z-a} \qquad |z| > |a|$$

and in the case that $k = 1$ we obtain

$$\mathcal{Z}\big(a^n n u_n\big) = \frac{az}{(z-a)^2} \qquad |z| > |a|\,.$$

The z-transform of a sequence c and of the shifted sequence with nth element $c_{n-\ell}$ where $\ell \in \mathbb{Z}$ are related by

$$\begin{aligned}\mathcal{Z}(c_{n-\ell})(z) &= \sum_{n\in\mathbb{Z}} c_{n-\ell} z^{-n} \\ &= \sum_{n\in\mathbb{Z}} c_n z^{-(n+\ell)} \\ &= z^{-\ell} \sum_{n\in\mathbb{Z}} c_n z^{-n} = z^{-\ell} \mathcal{Z}c(z) \qquad z \in \operatorname{roc}_z c.\end{aligned} \tag{6.5.3}$$

This is called the **shifting** property of the z-transform.

The z-transform obeys a convolution property analogous to that of the Laplace transform. Let F and G be discrete-time systems with periods P and discrete impulse responses f and g. Let $Hx = FGx$ be the discrete-time system formed by the composition F and G and with domain $\operatorname{dom}_P f\, g$. As shown in Section 6.1 the discrete impulse response of H is the discrete convolution $f * g$. Recall from (4.3.3) that the transfer function of a composition of linear time invariant systems is given by the product of the transfer functions, that is,

$$\lambda H(s) = \lambda G(s) \lambda F(s) \qquad e^{sP} \in \operatorname{roc}_z f \cap \operatorname{roc}_z g.$$

For s such that $e^{sP} \in \text{roc}_z f \cap \text{roc}_z g$ we have

$$\mathcal{Z}f(e^{sP}) = \lambda F(s), \quad \mathcal{Z}g(e^{sP}) = \lambda G(s), \quad \mathcal{Z}(f * g)(e^{sP}) = \lambda H(s)$$

from which it follow that

$$\mathcal{Z}(f * g)(z) = \mathcal{Z}f(z)\mathcal{Z}g(z) \qquad z \in \text{roc}_z f \cap \text{roc}_z g.$$

That is, the z-transform of a convolution of sequences is the multiplication of the z-transforms of those sequences. The region of convergence of $\mathcal{Z}(f * g)$ is the intersection of the regions of convergence of $\mathcal{Z}f$ and $\mathcal{Z}g$. This is called the **convolution property** of the z-transform.

Let H be a stable discrete-time system with absolutely summable discrete impulse response h. Because h is absolutely summable we have that $\text{roc}_z h$ contains the complex unit circle, that is, $\text{roc}_z h$ contains all those complex numbers with magnitude equal to one. We have the following relationships between the transfer function, the spectrum, the discrete-time Fourier transform, and the z-transform of a stable discrete-time system H and its discrete impulse response h,

$$\lambda H(j2\pi f) = \Lambda H(f) = \mathcal{D}h(f) = \mathcal{Z}h(e^{2\pi jPf}).$$

6.6 Difference equations

We have previously shown that interesting systems are found by consideration of a linear differential equation with constant coefficients. We have used these systems to model electrical and mechanical devices (Chapter 2). We will find that interesting discrete-time systems are found by consideration of a linear **difference equation** with constant coefficients. That is, an equation relating two sequences c and d of the form

$$\sum_{\ell=0}^{m} a_\ell c_{n-\ell} = \sum_{\ell=0}^{k} b_\ell d_{n-\ell} \qquad n \in \mathbb{Z} \tag{6.6.1}$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are complex constants. For example, in Section 5.6 we found that the number of complex operations required to compute a Cooley-Tukey fast Fourier transform of size $N = 2^n$ was $C_N = C_{2^n} = d_n$ where the sequence d_n satisfied the equation (5.6.4)

$$2^{n+1} = d_n - 2d_{n-1} \qquad n \geq 0$$

and where $C_1 = d_0 = 0$. This is in the form of (6.6.1) if we suppose that $d_n = 0$ for $n \leq 0$ and put $c_n = 2^{n+1}u_{n-1}$.

Another example is the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots.$$

Each element of the Fibonnacci sequence after the second is given by the sum of the previous two elements. Letting d_n be the elements of the Fibonacci sequence so that $d_0 = 0, d_1 = 1, d_2 = 1, \dots$ and putting $d_n = 0$ for $n < 0$ we have

$$d_n - d_{n-1} - d_{n-2} = \delta_{n-1}.$$

This is in the form (6.6.1) if we put $c_n = \delta_{n-1}$.

In order to study difference equations it is useful to study the equation

$$\sum_{\ell=0}^{m} a_\ell T_{P\ell} x = \sum_{\ell=0}^{k} b_\ell T_{P\ell} y \tag{6.6.2}$$

that relates two signals x and y. Zemanian [1965, Sec. 9.5] calls (6.6.2) the **continuous variable case** of a linear difference equation with constant coefficients. If x and y are signals satisfying this equation then the samples of x and y at multiples of P satisfy (6.6.1). That is, if we define sequences c and d by $c_n = x(nP)$ and $d_n = y(nP)$ then c and d satisfy (6.6.1) whenever x and y satisfy (6.6.2).

Suppose that H is a linear shift-invariant system with the property that the response $y = Hx$ to input signal x is such that x and y satisfy (6.6.2). The response of H to the complex exponential signal e^{st} satisfies $He^{st} = \lambda H(s)e^{st}$. Substituting $x(t) = e^{st}$ and $y = \lambda H(s)e^{st}$ into (6.6.2) gives

$$\sum_{\ell=0}^{m} a_\ell T_{P\ell} e^{st} = \sum_{\ell=0}^{k} b_\ell T_{P\ell} (\lambda e^{st})$$

where, to simplify notation, we have written simply λ for $\lambda H(s)$ above. Since $T_{P\ell} e^{st} = e^{-sP\ell} e^{st}$ we find that

$$e^{st} \sum_{\ell=0}^{m} a_\ell e^{-sP\ell} = \lambda e^{st} \sum_{\ell=0}^{k} b_\ell e^{-sP\ell}$$

and rearranging we find that the transfer function λH satisfies

$$\lambda H(s) = \frac{\sum_{\ell=0}^{m} a_\ell e^{-sP\ell}}{\sum_{\ell=0}^{k} b_\ell e^{-sP\ell}} = \frac{\sum_{\ell=0}^{m} a_\ell z^{-\ell}}{\sum_{\ell=0}^{k} b_\ell z^{-\ell}} = z^{k-m} \frac{\sum_{\ell=0}^{m} a_\ell z^{m-\ell}}{\sum_{\ell=0}^{k} b_\ell z^{k-\ell}}$$

where $z = e^{sP}$. Suppose that h is a sequence with z-transform

$$\mathcal{Z}h(z) = \lambda H(s) = z^{k-m} \frac{\sum_{\ell=0}^{m} a_\ell z^{m-\ell}}{\sum_{\ell=0}^{k} b_\ell z^{k-\ell}}.$$

It follows from (6.5.1) that H is a discrete-time system with discrete impulse response h. By applying the inverse z-transform we can find an explicit expression for h. This procedure is similar to how the impulse response of a system described by a differential equation was found by application of the inverse Laplace transform in Section 4.6. In the case that $m > k$ the term z^{k-m} can be incorporated into the denominator obtaining

$$\mathcal{Z}h(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^m + b_1 z^{k-1} + \cdots + b_k z^{m-k}}$$

and in the case that $m < k$ the term z^{m-k} can be incorporated into the numerator obtaining

$$\mathcal{Z}h(z) = \frac{a_0 z^k + a_1 z^{k-1} + \cdots + a_m z^{k-m}}{b_0 z^k + b_1 z^{k-1} + \cdots + b_k}.$$

In either case the order of the polynomials on the numerator and denominator are the same, that is, the order is $w = \max(m, k)$.

By factorising the polynomials on the numerator and denominator we obtain

$$\mathcal{Z}h(z) = \frac{a_0}{b_0} \frac{(z - \alpha_0)(z - \alpha_1) \cdots (z - \alpha_w)}{(z - \beta_0)(z - \beta_1) \cdots (z - \beta_w)}$$

where $\alpha_0, \dots, \alpha_w$ are the roots of the numerator polynomial and $\beta_0, \dots, \beta_w$ are the roots of the denominator polynomial. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\mathcal{Z}h(z) = \frac{a_0}{b_0} \frac{(z - \alpha_d)(z - \alpha_{d+1}) \cdots (z - \alpha_w)}{(z - \beta_d)(z - \beta_{d+1}) \cdots (z - \beta_w)}, \tag{6.6.3}$$

where d is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \quad \alpha_1 = \beta_1, \quad \dots, \quad \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator $\alpha_d, \dots, \alpha_w$ are called the **zeros** and the roots from the denominator $\beta_d, \dots, \beta_w$ are called the **poles**. For a discrete-time system, the number of poles and zeros are equal. A pole-zero plot is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero (Figure 6.8).

The z-transform pair (6.5.2) has the term z on its numerator and so it is convenient to write

$$\mathcal{Z}h(z) = \frac{a_0}{b_0} z \frac{(z - \alpha_d)(z - \alpha_{d+1}) \cdots (z - \alpha_w)}{z(z - \beta_d)(z - \beta_{d+1}) \cdots (z - \beta_w)}.$$

Applying partial fraction to polynomial quotient yields

$$\mathcal{Z}h(z) = \frac{a_0}{b_0} z \sum_{\ell \in K} \frac{A_\ell}{(z - \beta_\ell)^{r_\ell}}$$

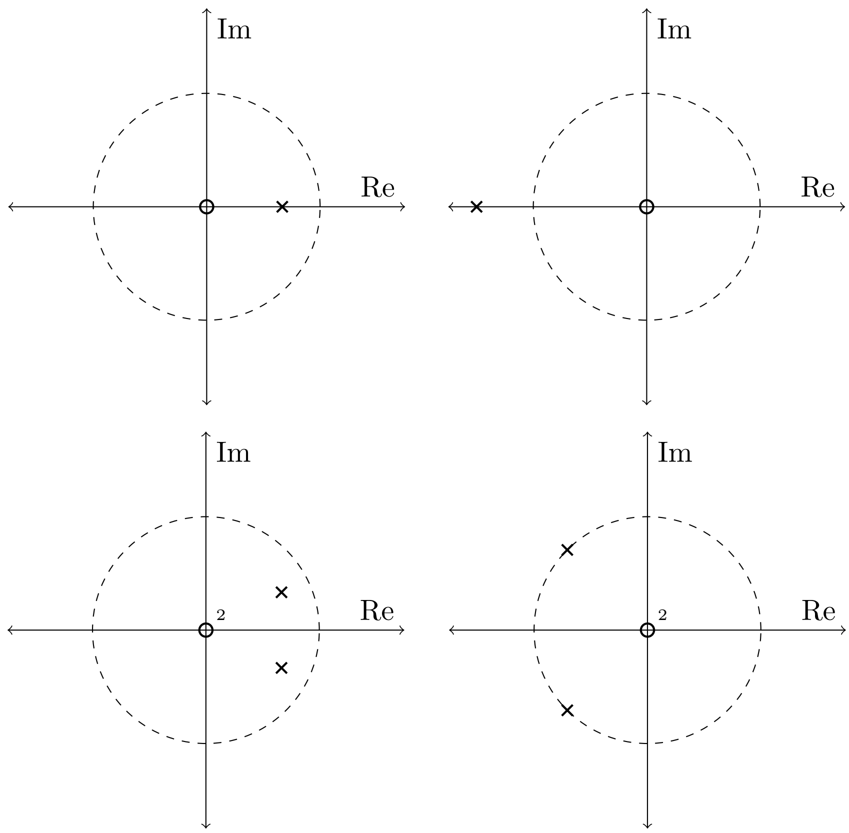

Figure 6.8: Pole zero plots for discrete-time systems corresponding with first order difference equations $c_n = d_n - \frac{2}{3}d_{n-1}$ (top left) and $c_n = d_n + \frac{3}{2}d_{n-1}$ (top right) and second order difference equations $z^2 - \frac{4}{3}x - \frac{4}{9}$ (bottom left) and $z^2 + \sqrt{2}z + 1$ (bottom right). The plots of the left correspond with stable systems because all poles are contained inside the complex unit circle (dashed). Plots on the right correspond with unstable systems because there exist poles on or outside the complex unit circle. The small 2's above the zero on the lower plots indicate the existence of two zeros at the origin.

where r_ℓ are positive integers, A_ℓ are complex constants, and K is a subset of the indices from $\{d, d+1, \dots, w\}$. We need to consider those terms where $\beta_\ell = 0$ separately. Let K_1 be the subset of indices from K such that $\beta_\ell = 0$ when $\ell \in K_1$ and let K_2 be the subset such that $\beta_\ell \neq 0$ when $\ell \in K_2$. Now

$$\mathscr{Z}h(z) = \frac{a_0}{b_0} \sum_{\ell \in K_1} \frac{A_\ell}{z^{r_\ell - 1}} + \sum_{\ell \in K} B_\ell \frac{\beta_\ell^{r_\ell - 1}(r_\ell - 1)! z}{(z - \beta_\ell)^{r_\ell}}.$$

where

$$B_\ell = \frac{a_0 A_\ell}{b_0 \beta_\ell^{r_\ell - 1}(r_\ell - 1)!}.$$

Those terms of the form $A_\ell z^{1 - r_\ell}$ correspond with sequences $A_\ell \delta_{n + r_\ell - 1}$ where δ is the delta sequence. From (6.5.2) with $k = r_\ell - 1$ those terms of the form

$$\frac{\beta_\ell^{r_\ell - 1}(r_\ell - 1)! z}{(z - \beta_\ell)^{r_\ell}}$$

are found to correspond with sequences $B_\ell \beta_\ell^n [n]_{r_\ell - 1} u_n$ where u is the step sequence. Other sequences with the same z-transform are disregarded because they are not right sided and so do not correspond with a causal discrete-time system. Combing the above results we find that the discrete impulse response h of the discrete-time system H takes the form

$$h_n = \frac{a_0}{b_0} \sum_{\ell \in K_1} A_\ell \delta_{n + r_\ell - 1} + \sum_{\ell \in K_2} B_\ell \beta_\ell^n [n]_{r_\ell - 1} u_n.$$

The discrete impulse response is absolutely summable only if the poles satisfy $|\beta_\ell| < 1$ for all $\ell = d, \dots, w$ as a result of the terms β_ℓ^n that occur when $\beta_\ell \neq 0$. The system H is stable if and only if h is absolutely summable (Exercise 6.8) and so a discrete-time system is stable if and only if no poles lie outside or on the complex unit circle.

We now consider some specific examples of difference equations and their corresponding discrete-time systems. Consider the difference equation

$$c_n = d_n - a d_{n-1} \qquad n \in \mathbb{Z} \tag{6.6.4}$$

where $a \in \mathbb{C}$. This is called a **first order difference equation**. Suppose that H is a discrete-time system such that the response $y = H(x)$ to input x satisfies

$$x = y - a T_P(y).$$

The transfer function of H is

$$\lambda(H, s) = \frac{1}{1 - a e^{-sP}} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

where $z = e^{sP}$. The system has a single zero at $z = 0$ and a single pole at $z = a$. The system will be stable if and only if this pole lies strictly inside the complex unit circle, that is, if and only if $|a| < 1$. The discrete impulse response is found to be $h_n = a^n u_n$ by putting $k = 0$ in (6.5.2). Other sequences with this z-transform are discarded because they do not correspond with a causal system. When $|a| < 1$ the region of convergence contains the unit circle and the system has spectrum

$$\Lambda(H, f) = \lambda(H, j2\pi f) = \mathcal{Z}(h, e^{2\pi jPf}) = \frac{e^{2\pi jPf}}{e^{2\pi jPf} - a}.$$

The magnitude and phase spectrum are plotted in Figure 6.9 in the case that $a = \frac{1}{2}$ and $\frac{1}{10}$.

Now consider the difference equation

$$c_n = d_n - ad_{n-1} - bd_{n-2} \qquad n \in \mathbb{Z}.$$

where $a, b \in \mathbb{C}$. This is called a **second order difference equation**. Suppose that H is a discrete-time system with reponse $y = H(x)$ satisfying the equation $x = y - aT_P(y) - bT_{2P}(y)$. The transfer function is

$$\lambda(H) = \frac{1}{1 - ae^{-sP} - be^{-2sP}} = \frac{z^2}{z^2 - az - b} = \mathcal{Z}(h)$$

where h is the discrete impulse response of H. The system has two zeros at $z = 0$ and two poles given by the roots of the polynomial $z^2 - az - b$. The z-transform can be inverted to obtain h (Exercise 6.5). The system H is stable if and only if both poles lie strictly inside the complex unit circle (Figure 6.8). In this case, H has spectrum

$$\Lambda(H, f) = \mathcal{Z}(h, e^{2\pi jPf}) = \frac{e^{2\pi jPf}}{e^{4\pi jPf} - ae^{2\pi jPf} - b}.$$

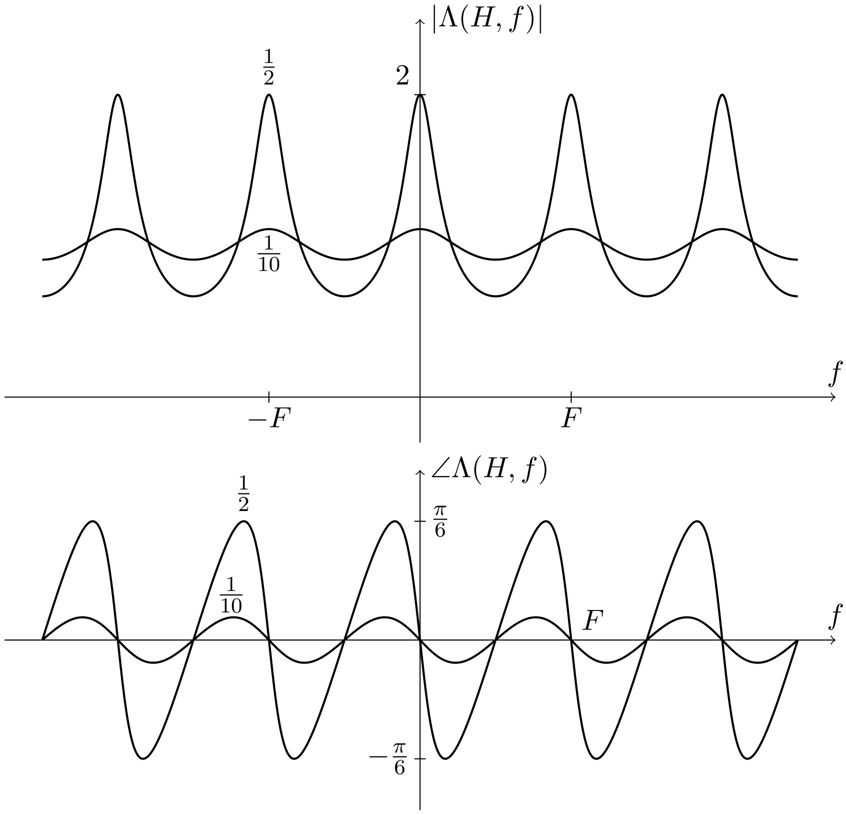

Figure 6.9: Magnitude and phase spectrum of the discrete-time H with discrete implulse response $h_n = a^n u_n$ for $a = \frac{1}{2}$ and $\frac{1}{10}$ and period $P = \frac{1}{F}$. The spectrum is periodic with period $F = \frac{1}{P}$. This system corresponds with the first order difference equation $c_n = d_n - 2d_{n-1}$.

Exercises

6.1. Let x be the signal with Fourier transform

$$\hat{x}(t) = \tfrac{4}{3}\Pi(f) - \cos(2\pi f)\big(\Pi(2f-3) + \Pi(2f+3)\big).$$

Plot the Fourier transform. Find and plot x.

6.2. Find the Fourier transform of the Blackman window (6.4.1).

6.3. Show that the z-transform of the sequence $a^n u_n$ is $z/(z-a)$ with region of convergence $|z| > |a|$.

6.4. Show that the z-transform of the sequence $[n]_k u_n$ where $[n]_k = n(n-1)\dots(n-k+1)$ is a falling factorial is

$$\mathscr{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1.$$

6.5. Find the discrete impulse response of the discrete time system corresponding with the second order difference equation $c_n = d_n - ad_{n-1} - bd_{n-2}$.

6.6. Let d_n be a sequence satisfying $d_n = 2d_{n-1} + 2^{n+1}$ and suppose that $d_0 = 0$. Show that $d_n = 2^{n+1}n$ for $n = 1, 2, \dots$.

6.7. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ satisfies the recursive equation $d_0 = 0, d_1 = 1$, and $d_n = d_{n-1} + d_{n-2}$ for $n \geq 2$. Find a closed form expression for the nth Fibonacci number.

∗6.8. Show that a discrete time system is stable if and only if its discrete impulse response is absolutely summable.

∗6.9. Let f and g be absolutely summable sequences. Show that the discrete convolution $f * g$ is also absolutely summable.

∗6.10. Let H be a discrete time system with discrete impulse response h. The set $\operatorname{roc}_z h$ is defined as those complex numbers $z = e^{sP}$ such that $s = \operatorname{cep}\operatorname{dom}_P h$. Show that $\operatorname{roc}_z h$ is precisely the set of nonzero complex numbers such that the sequence $h_n z^{-n}$ is absolutely summable.

∗6.11. Let f, g, h be complex valued sequences such that

$$\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |f_k h_m g_{n-m-k}| < \infty.$$

Show that the discrete convolution is associative for these sequences. That is, show that $(f * g) * h = f * (g * h)$.

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