Testable linear shift-invariant systems (Exercise Solutions)

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Contents

Chapter 1

Signals and systems

Exercises

1.1. How many distinct functions from the set $X = \{\text{Mario}, \text{Link}\}$ to the set $Y = \{\text{Freeman}, \text{Ryu}, \text{Sephiroth}\}$ exist? Write down each function, that is, write down all functions from the set $X \to Y$.

Solution: Each of the two elements in X can be mapped to one of the three elements of Y. There are thus $3^2 = 9$ distinct functions in $X \to Y$. They are

$$f_1(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_2(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_3(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_4(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

$$f_5(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_6(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_7(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases} \qquad f_8(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_9(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

1.2. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal. **Solution:** The magnitude of u is less than or equal to one and so the signal is bounded. The signal is not periodic, since for any hypothesised period $T > 0$ we have $u(T) = 1$ but $u(0) = 0$. The signal is not absolutely integrable, nor an energy signal since

$$\|u\|_1 = \|u\|_2 = \int_{-\infty}^{\infty} |u(t)|\, dt = \int_0^{\infty} dt$$

is not finite.

1.3. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

Solution: For any a and b

$$\int_a^b t^2 dt = \frac{b^3}{3} - \frac{a^3}{3}$$

is finite and so t^2 is locally integrable. Put $a = 0$ and $b > 0$ and

$$\int_0^b \frac{1}{t^2} dt = -\frac{1}{b} + \lim_{t \to 0} \frac{1}{t} = \infty.$$

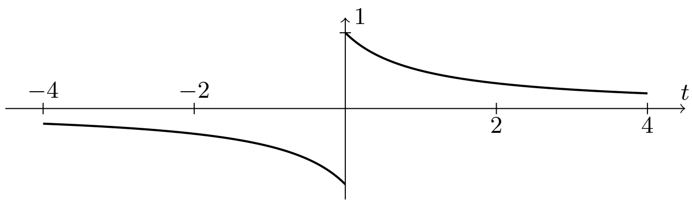
The limit above diverges and so $\frac{1}{t^2}$ is not locally integrable.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \le 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

Solution:



The signal is bounded since $|x(t)| < M$ for any $M > 1$. The signal is locally integrable because it is bounded, i.e., for any finite constants a and b

$$\int_a^b |x(t)|\, dt < \int_a^b M dt = (b-a)M < \infty.$$

The signal x is not absolutely integrable since

$$\begin{aligned} \|x\|_1 &= \int_{-\infty}^{\infty} |x(t)|\, dt \\ &= 2\int_0^{\infty} \frac{1}{t+1} dt \\ &= 2\int_1^{\infty} \frac{1}{t} dt \\ &= 2\log(1) + \lim_{t\to\infty} 2\log(t) \end{aligned}$$

and the limit diverges. The signal is square integrable since

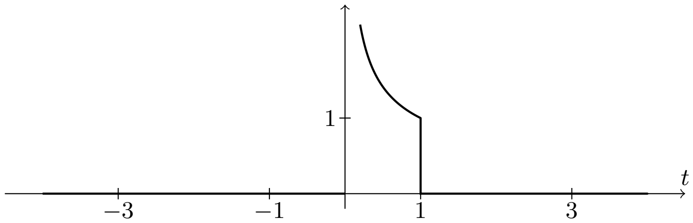
$$\begin{aligned} \|x\|_2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\int_0^{\infty} \frac{1}{(t+1)^2} dt \\ &= 2\int_1^{\infty} \frac{1}{t^2} dt \\ &= 2 - \lim_{t\to\infty} \frac{2}{t} = 2. \end{aligned}$$

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

Solution:



The integral

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1/2} dt = [2\sqrt{t}]_0^1 = 2$$

and so x is absolutely integrable. The integral

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1} dt = [\log(t)]_0^1 = \log(1) - \lim_{t \to 0} \log(t) = \infty$$

and so x is not square integrable.

1.6. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 5 and a change of variables). **Solution:** From (1.1.4) we the energy of e^{-t^2} is $\sqrt{\pi}$. Now

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \frac{1}{\alpha} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{\alpha}$$

by the change of variables $\tau = \alpha t$.

1.7. Show that the signal t^2 is differentiable, but the step function u and rectangular pulse Π are not. **Solution:** We have

$$\lim_{h \to 0} \frac{(t+h)^2 - t^2}{h} = \lim_{h \to 0} \frac{2th + h^2}{h} = 2t.$$

$$\lim_{h \to 0} \frac{t^2 - (t-h)^2}{h} = \lim_{h \to 0} \frac{2th - h^2}{h} = 2t$$

and so t^2 is continuously differentiable with derivative $\frac{d}{dt} t^2 = 2t$. At $t = 0$ the corresponding limits for the step function are

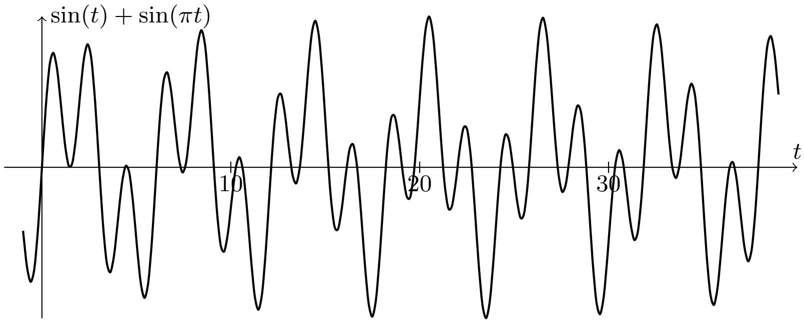
$$\lim_{h \to 0} \frac{u(h) - u(0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

but

$$\lim_{h \to 0} \frac{u(0) - u(-h)}{h} = \lim_{h \to 0} \frac{1}{h} = \infty$$

so the step function u is not differentiable at $t = 0$. A similar argument at $t = \frac{1}{2}$ or $t = -\frac{1}{2}$ shows that Π is not differentiable.

1.8. Plot the signal $\sin(t) + \sin(\pi t)$. Show that this signal is not periodic. **Solution:** A plot of the signal is below:



The following argument is due to Qiaochu Yuan. Suppose $\sin(t)+\sin(\pi t)$ is periodic. Then

$$\sin(t) + \sin(\pi t) = \sin(t + T) + \sin(\pi t + T)$$

for some $T > 0$. Differentiating both sides twice with respect to t gives

$$\sin(t) + \pi^2 \sin(\pi t) = \sin(t + T) + \pi^2 \sin(\pi t + T)$$

Subtracting the first equation from the second gives $\sin(t) = \sin(t + T)$ and substituting this into the second equation gives $\sin(\pi t) = \sin(\pi t + T)$. The equation $\sin(t) = \sin(t + T)$ implies that $T = 2\pi k$ for some integer $k \neq 0$. The equation $\sin(\pi t) = \sin(\pi t + T)$ implies that $T = 2\ell$ for some integer $\ell \neq 0$. We would thus have $2\pi k = 2\ell$ and so $\pi = \frac{\ell}{k}$. However, this is impossible because π is irrational. Thus $\sin(t) + \sin(\pi t)$ is not periodic.

1.9. Show that the set of locally integrable signals L_{loc}, the set of absolutely integrable signals L^1, and the set of square integrable signals L^2 are linear shift-invariant spaces. **Solution:** Let $x, y \in L^1$ and $a, b \in \mathbb{C}$. Now

$$\begin{aligned}
\|ax + by\|_1 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|\, dt \\
&\leq \int_{-\infty}^{\infty} a\,|x(t)| + b\,|y(t)|\, dt \qquad \text{triangle inequality} \\
&= a\|x\|_1 + b\|y\|_1 < \infty
\end{aligned}$$

and so $ax + by \in L_1$ and L_1 is a linear space. Also

$$\begin{aligned}
\|T_\tau x\|_1 &= \int_{-\infty}^{\infty} |T_\tau x(t)|\, dt \\
&= \int_{-\infty}^{\infty} |x(t - \tau)|\, dt \\
&= \int_{-\infty}^{\infty} |x(k)|\, dk \qquad \text{change variable } k = t - \tau \qquad = \|x\|_1 < \infty
\end{aligned}$$

and so L_1 is a shift-invariant space.

Now

$$\begin{aligned}
\|ax + by\|_2^2 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|^2\, dt \\
&\int_{-\infty}^{\infty} |ax(t)|^2 + |by(t)|^2 + 2\operatorname{Re}\big(a^* x(t)^* b y(t)\big) dt
\end{aligned}$$

where $*$ denotes the complex cojugate and Re denotes the real part of a complex number. Now

$$\operatorname{Re}\big(a^* x(t)^* b y(t)\big) \leq |ax(t)|\,|by(t)| \leq \max(|ax(t)|^2, |by(t)|^2) \leq |ax(t)|^2 + |by(t)|^2$$

and so

$$\begin{aligned}\|ax + by\|_2^2 &\le \int_{-\infty}^{\infty} 3\,|ax(t)|^2 + 3\,|by(t)|^2\,dt \\ &= \int_{-\infty}^{\infty} 3\,|a|^2\,|x(t)|^2 + 3\,|b|^2\,|y(t)|^2\,dt \\ &= 3\,|a|^2\,\|x\|_2^2 + 3\,|b|^2\,\|y\|_2^2 < \infty\end{aligned}$$

and L_2 is thus a linear space. Also

$$\|T_\tau x\|_2^2 = \int_{-\infty}^{\infty} |T_\tau x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t-\tau)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \|x\|_2^2 < \infty$$

and so L_2 is a shift-invariant space.

1.10. Show that the set of periodic signals is a shift-invariant space, but not a linear space. **Solution:** Let P be the set of periodic signals. If $x \in P$ then there exists $T > 0$ such that $x(t + kT) = x(t)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. The shifted signal $T_\tau x \in P$ since, for the same T, we have

$$T_\tau x(t - kT) = x(t - \tau - kT) = x(t - \tau) = T_\tau x(t)$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$. Since $x \in P$ and $\tau \in \mathbb{R}$ are arbitrary, this holds for all signals $x \in P$ and all shifts $\tau \in reals$. Thus, the set of periodic signals is a shift invariant space.

The set of period signals is not a linear space. Consider the signal $x(t) = \sin(t)$ with period 2π and $y(t) = \sin(\pi t)$ with period 2. Both x and y are in P. However, exercise 1.8 shows that the sum $x(t) + y(t) = \sin(t) + \sin(\pi t)$ is not periodic, that is, $x + y \in P$.

1.11. Show that the set of bounded signals is a linear shift-invariant space. **Solution:** Let B be the set of bounded signals. If $x \in B$ there exists $M > 0$ such that $|x(t)| < M$ for all $t \in \mathbb{R}$ then the shift $T_\tau x(t)$ satisfies $|T_\tau x(t)| < M$ for all $t \in \mathbb{R}$. Since x and τ are arbitrary this holds for all $x \in B$ and $\tau \in \mathbb{R}$. Thus B is a shift invariant space.

Let $x \in B$ and $y \in B$ be bounded signals. There exists $M_x > 0$ and $M_y > 0$ such that

$$|x(t)| < M_x \qquad |y(t)| < M_y \qquad \text{for all } t \in \mathbb{R}.$$

Now for $a, b \in \mathbb{C}$ the signal $ax + by$ satisfies

$$|ax(t) + by(t)| \le |a|\,|x(t)| + |b|\,|y(t)| < |a|\,M_x + |b|\,M_y$$

for all $t \in \mathbb{R}$. Thus the linear combination $ax + by$ is bounded. Since $a, b \in \mathbb{C}$ and $x, y \in B$ are arbtirary this holds for all $a, b \in \mathbb{C}$ and all $x, y \in B$ and so B is a linear space.

1.12. Let $K > 0$ be a fixed real number. Show that the set of signals bounded below K is a shift invariant space, but not a linear space. **Solution:** Let B_K be the set of signals bounded less than K, that is,

$$B_K = \{x \in \mathbb{R} \to \mathbb{C}\ ;\ |x(t)| < K \text{for all } t \in \mathbb{R}\}.$$

If $x \in B_K$ then $|T_\tau x(t)| < K$ for all $t \in \mathbb{R}$ and so B_K is $T_\tau x \in B_K$. Thus, B_K is a shift invariant space.

Consider constant signals $x(t) = K/2$ and $y(t) = 2K/3$. Both x and y are bounded less than K and so are in B_K However, the signal $x + y$ is such that

$$|x(t) + y(t)| = K/2 + 2K/3 = 7K/6 > K$$

and so $x + y \notin B_K$. Thus B_K is not a linear space.

1.13. Show that the set of even signals and the set of odd signals are not shift invariant spaces.

1.14. Show that the integrator I_c with finite $c \in \mathbb{R}$ is not stable. **Solution:** Put $M > 1$. The shifted step function $u(t + a)$ is locally integrable and bounded below M, i.e. $|u(t + a)| \leq 1 < M$ for all $t \in \mathbb{R}$. However, the response of the integrator I_a to $u(t + a)$ is

$$I_a u(t+a) = \int_{-a}^{t} u(\tau + a) d\tau = \begin{cases} \int_{-a}^{t} d\tau = t + a & t \geq -a \\ 0 & t < -a \end{cases},$$

and this is not a bounded signal, that is, for every K we have $t + a > K$ whenever $t > K - a$.

1.15. Show that if the signal x is locally integrable and $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ then $I_\infty x(t) = \int_{-\infty}^{t} x(t)dt < \infty$ for all $t \in \mathbb{R}$. **Solution:** We have

$$\begin{aligned} I_\infty x(t) \leq |I_\infty x(t)| &= \left| \int_{-\infty}^{t} x(t)dt \right| \\ &\leq \int_{-\infty}^{t} |x(t)|\, dt \\ &= \int_{-\infty}^{0} |x(t)|\, dt + \int_{0}^{t} |x(t)|\, dt \end{aligned}$$

Now $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ by assumption and $\int_{0}^{t} |x(t)|\, dt$ because x is locally integrable. It follows that

1.16. Show that the integrator I_∞ is not stable. **Solution:** By default the domain for I_∞ is the subset of locally integrable signals for which $\int_{-\infty}^{0} |x(t)|\, dt < \infty$. The step function $u(t)$ is in this domain. The argument now follows similiarly to Exercise 1.16.

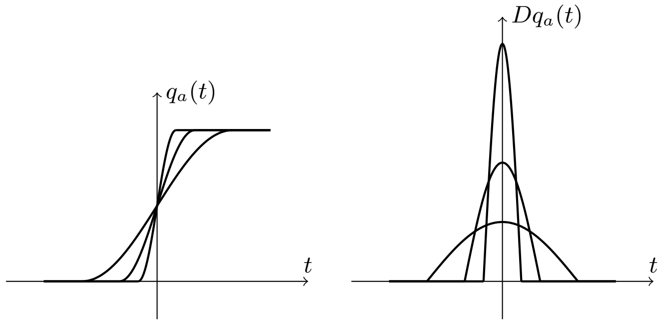
1.17. Show that the differentiator system D is not stable. **Solution:** Put $M > 2$. Define the signal

$$q_a(t) = \begin{cases} 0 & 2t < -a \\ 1 + \sin\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 2 & 2t > a, \end{cases}$$

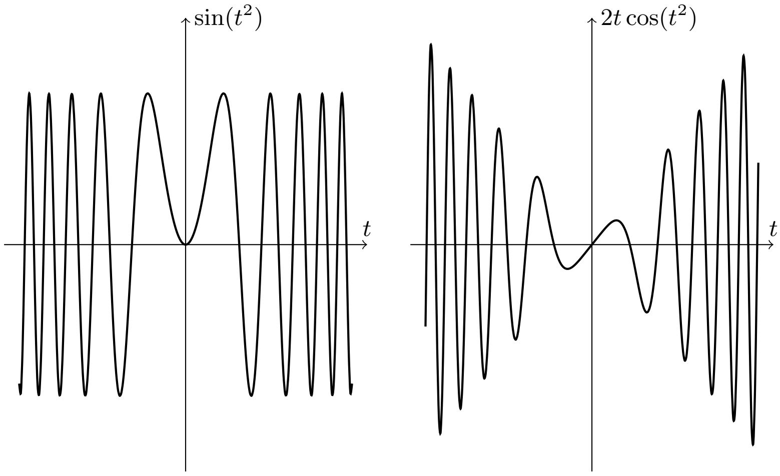
and observe that q_a is differentiable and bounded below M. The response of the differentiator D to q_a is

$$Dq_a(t) = \begin{cases} 0 & 2t < -a \\ \frac{\pi}{a} \cos\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 1 & 2t > a. \end{cases}$$

The signal p_a and the response Dp_a are plotted below for $a = \frac{1}{2}, 1$ and 2. The response Dp_a obtains a maximum amplitude of $\frac{\pi}{a}$ at $t = 0$. So D is not stable because for any K we can choose $a < \frac{\pi}{K}$ so that $\frac{\pi}{a} > K$.



Another solution was suggested by Badri Vellambi. Consider the signal $x(t) = \sin(t^2)$ plotted in the figure below. This signal is bounded below any $M > 1$. The response of the differentiator is $Dx(t) = 2t\cos(t^2)$ and this is not bounded.



1.18. Show that the shifter T_τ is linear and shift-invariant and that the time-scaler is linear, but not time invariant. **Solution:** The shifter T_τ is shift-invariant since

$$T_k T_\tau x = T_k x(t - \tau) = x(t - \tau - k) = T_\tau x(t - k) = T_\tau T_k x$$

for all signals x, that is, shifters commute with shifters. The shifter is linear because

$$T_\tau(ax + by) = ax(t - \tau) + by(t - \tau) = aT_\tau x + bT_\tau y.$$

The time-scaler $Hx = x(\alpha t)$ is linear because

$$H(ax + by) = ax(\alpha t) + by(\alpha t) = aHx + bHy.$$

The system is not shift-invariant because

$$HT_\tau x = Hx(t - \tau) = x(\alpha t - \tau)$$

but

$$T_\tau H x = T_\tau x(\alpha t) = x(\alpha(t-\tau)) = x(\alpha t - \alpha\tau),$$

and these signals are not equal in general. For example consider the rectangular pulse Π. With time-scaling parameter $\alpha = 2$ and shift $\tau = 1$,

$$HT_1\Pi = \Pi(2t-1) \neq \Pi(2t-2) = T_1 H\Pi.$$

1.19. Show that the integrator I_c with finite $c \in \mathbb{R}$ is linear, but not shift-invariant. **Solution:** The system is linear because, if $x, y \in L_{\text{loc}}$, then

$$\begin{aligned} I_c(ax+by) &= \int_{-c}^{t} ax(\tau) + by(\tau) d\tau \\ &= a\int_{-c}^{t} x(\tau)d\tau + b\int_{-c}^{t} y(\tau)d\tau \\ &= aI_c x + bI_c y. \end{aligned}$$

The system is not shift-invariant because

$$T_k I_c x = I_c(x, t-k) = \int_{-c}^{t-k} x(\tau)d\tau$$

but

$$I_c T_k x = \int_{-c}^{t} x(\tau - k)d\tau.$$

We now need only find some signal $x \in L_{\text{loc}}$ for which the integrals on the right hand side of the above equations are not equal. Choose the signal $x = 1$, i.e., the signal that is equal to 1 for all time. In this case

$$T_k I_c 1 = \int_{-c}^{t-k} d\tau = t - k + c \neq t + c = \int_{-c-k}^{t-k} d\tau = I_c T_k 1 \qquad \text{when } k \neq 0.$$

1.20. Show that the integrator I_∞ is linear and shift-invariant. **Solution:** The system is linear because

$$\begin{aligned} I_\infty(ax+by) &= \int_{-\infty}^{t} ax(\tau) + by(\tau) d\tau \\ &= a\int_{-\infty}^{t} x(\tau)d\tau + b\int_{-\infty}^{t} y(\tau)d\tau \\ &= aI_\infty x + bI_\infty y. \end{aligned}$$

The system is shift-invariant because

$$T_k I_\infty x = I_\infty x(t-k) = \int_{-\infty}^{t-k} x(\tau)d\tau,$$

and

$$I_\infty T_k x = \int_{-\infty}^{t} x(\tau - k)d\tau = \int_{-\infty}^{t-k} x(\tau)d\tau.$$

1.21. State whether the system $Hx = x + 1$ is linear, shift-invariant, stable. **Solution:** It is not linear because for any signal x and real number $a \neq 1$,

$$H(ax) = ax + 1 \neq aHx = a\big(x + 1\big) = ax + a.$$

It is shift-invariant because

$$HT_\tau x = x(t - \tau) + 1 = T_\tau(x + 1) = T_\tau Hx.$$

It is stable because for any signal x with $x(t) < M$ for all $t \in \mathbb{R}$,

$$Hx(t) = x(t) + 1 < M + 1 \qquad \text{for all } t \in \mathbb{R}.$$

1.22. State whether the system $Hx = 0$ is linear, shift-invariant, stable. **Solution:** It is linear because

$$H(ax + by) = 0 = aHx + bHy = 0.$$

It is shift-invariant because

$$HT_\tau x(t) = 0 = Hx(t - \tau).$$

It is stable because for any $K > 0$,

$$Hx(t) = 0 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.23. State whether the system $Hx = 1$ is linear, shift-invariant, stable. **Solution:** It is not linear because for any signal x and real number $a \neq 1$

$$H(ax) = 1 \neq aHx = a.$$

It is shift-invariant because

$$HT_\tau x = 1 = T_\tau(1) = T_\tau Hx.$$

It is stable because for any $K > 1$,

$$|Hx(t)| = 1 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.24. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is neither absolutely integrable nor square integrable. **Solution:** This is plain and does not really require further explanation, but I've found some students desire more rigour.

Since x does not equal to zero almost everywhere there exist some finite real numbers a and b such that $\int_a^b |x(t)|\,dt = C > 0$. Let k be an integer such $-kT < a$ and $kT > b$ so that the integral over $2k + 1$ periods

$$\int_{-kT}^{kT} |x(t)|\,dt \geq \int_a^b |x(t)|\,dt = C > 0.$$

Now, since x has period T

$$\int_{-ckT}^{ckT} |x(t)|\,dt = (2c + 1)\int_{-kT}^{kT} |x(t)|\,dt \geq (2c + 1)C > 0$$

for integers c and since this integral is increasing monotonically with c we have $\int_{-ckT}^{ckT} |x(t)|\,dt \geq \lfloor 2c+1 \rfloor C$ for all $c \in \mathbb{R}$ where $\lfloor 2c+1 \rfloor$ denotes the largest integer less than or equal to $2c+1$. Now,

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\,dt = \lim_{c\to\infty} \int_{-ckT}^{ckT} |x(t)|\,dt \geq \lim_{c\to\infty} \lfloor 2c+1 \rfloor C = \infty,$$

and so, x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Maxima, Sage, Mathematica, or Maple). **Solution:** We provide two solutions. Let v_i, v_o, v_1 and v_2 be the voltages over the input resistor R_i, the output resistor R_o, and resistors R_1 and R_2 respectively. We have 8 unknown voltages $x, y, v_i, v_1, v_2, v_o, v_+, v_-$. We will need 7 independent equations to find an equation relating x and y. All currents are considered to be flowing either downwards or to the right in the circuit diagram. The first 4 equations are given by voltages over each resitor,

$$\begin{aligned} x &= v_- + v_1 \\ v_- &= y + v_2 \\ v_- &= v_+ + v_i \\ y &= v_o + A(v_+ - v_-) \end{aligned}$$

The next two equations apply Kirchoff's current law to each node betweeen resistors. The currents into the 3 way connection between R_i, R_1 and R_2 sum to zero, and so

$$\frac{v_1}{R_1} = \frac{v_2}{R_2} + \frac{v_i}{R_i}$$

by Ohm's law. Finally the currents through R_o and R_2 are the same, and so

$$\frac{v_o}{R_o} = \frac{v_2}{R_2}.$$

The final equation simply observes that the non-inverting terminal v_+ is connected to ground

$$v_+ = 0.$$

We now have 7 linearly independent equations for the 8 unknowns $x, y, v_i, v_1, v_2, v_o, v_+, v_-$. We can use these to find an equation that describes y in terms of x. The Mathematica command

```
Simplify[Solve[{x == vm + v1,
   vm == y + v2,
   vm == vp + vi,
   y == vo + A*(vp - vm),
   v1/r1 == vi/ri + v2/r2,
   vo/ro == v2/r2,
   vp == 0,
   r1 > 0, r2 > 0, ro > 0, ri > 0, A > 0},
  {y,vi,vo,v2,v1,vp,vm}, Reals]]
```

or Maxima command

```
linsolve([x=vm+v1,
          vm=y+v2,
          vm=vp+vi,
          y=vo+A(vp-vm),
          v1/R1=v2/R2+vi/Ri,
          v2/R2=vo/Ro,
          vp=0],
          [y,vp,vm,v1,v2,vo,vi]);
```

readily obtains

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x.$$

The second solution is thanks to Badri Vellambi. Badri sets $v_i = v_+ - v_-$ so that the voltage over the dependent voltage source is Av_i. Consider the operational amplifier circuit with feedback presented in Fig. 2.1. Suppose that the voltage signal fed into the circuit is $x(t)$ and the voltage signal measured at the output of the opamp is $y(t)$.

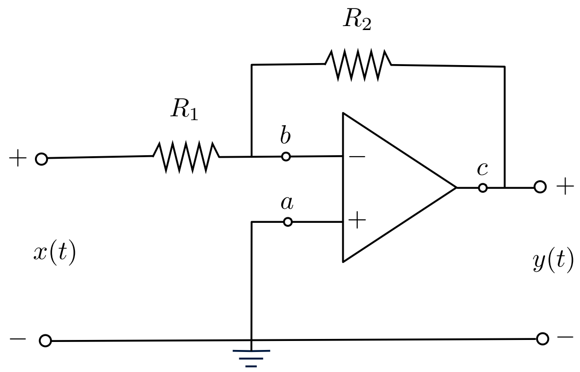


Figure 2.1: The circuit

To simplify the circuit, one has to use the model for the opamp given in Fig. 2.2 which involves the voltage-controlled voltage-source (VCVS) at the output side (indicated in green). While replacing the operational amplifier with its model, it must be noted that the positive terminal of the operational amplifier is connected to the ground.

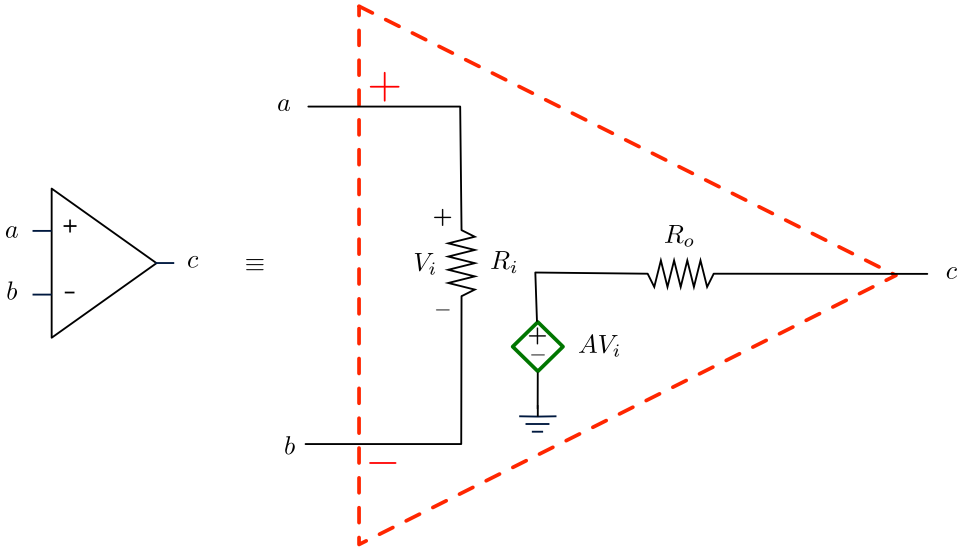


Figure 2.2: The model for an operational amplifier

Upon replacement, we obtain the following equivalent circuit. Again notice that since the positive terminal of the opamp was connected to the ground, the voltage output by the VCVS is AV_i where V_i is the voltage between the ground and the top of the resistance R_i, and is measured against the flow of the current $i - i_1$ as is indicated in the figure.

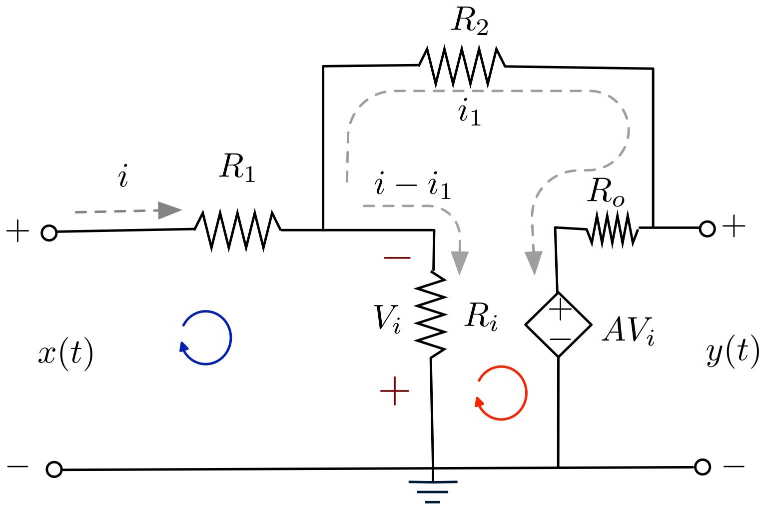


Figure 2.3: The operational amplifier circuit with the model

Applying Kirchoff's law to the outer loop indicated in blue in Fig. 2.3, we obtain the following equation.

$$x(t) = iR_1 + (i - i_1)R_i = i(R_1 + R_i) - i_1 R \tag{2.0.1}$$

Note that by definition, the voltage V_i that controls the VCVS is the voltage across R_i measured against the indicated direction of the current $i - i_1$, and is given by

$$V_i = -(i - i_1)R_i. \tag{2.0.2}$$

Next, writing out the Kirchoff's law for the inner loop indicated in red, we obtain the following.

$$0 = i_1R_2 + i_2R_0 + AV_i - (i - i_1)R_i \tag{2.0.3}$$

Substituting V_i in the above equation with the RHS of (2.0.2), we obtain the following.

$$\begin{aligned} 0 &= i_1(R_2 + R_0) - A(i - i_1)R_i - (i - i_1)R_i && (2.0.4)\\ &= -i(1 + A)R_i + i_1((1 + A)R_i + R_0 + R_2) && (2.0.5)\end{aligned}$$

Combining (2.0.5) and (2.0.1), we obtain the following linear system of equations governing the electrical circuit.

$$\begin{bmatrix} R_1 + R_i & -R_i \\ -(1 + A)R_i & (1 + A)R_i + R_0 + R_2 \end{bmatrix} \begin{bmatrix} i \\ i_1 \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix} \tag{2.0.6}$$

Solving the above linear system, we identify the current in the different branches to be

$$\begin{bmatrix} i \\ i_1 \end{bmatrix} = x(t) \begin{bmatrix} \frac{(1+A)R_i+R_0+R_2}{(1+A)R_iR_1+R_0R_1+R_2R_1+R_0R_i+R_2R_i} \\ \frac{(1+A)R_i}{(1+A)R_iR_1+R_0R_1+R_2R_1+R_0R_i+R_2R_i} \end{bmatrix}. \tag{2.0.7}$$

Lastly, notice that

$$\begin{aligned} y(t) &= i_1R_0 + AV_i && (2.0.8)\\ &= i_1R_0 - (i - i_1)R_i. && (2.0.9)\end{aligned}$$

Substituting the solutions for i and i_1 in terms of $x(t)$, we obtain the following.

$$y(t) = \left(\frac{R_iR_0 - R_2R_iA}{(1 + A)R_iR_1 + R_0R_1 + R_2R_1 + R_0R_i + R_2R_i} \right) x(t) \tag{2.0.10}$$

2.2. Figure 2.5 depicts a mechanical system involving two masses, two springs, and a damper connected between two walls. Suppose that the spring K_2 is at rest when the mass M_2 is at position $p(t) = 0$. A force, represented by the signal f, is applied to mass M_1. Derive a differential equation relating the force f and the position p of mass M_2. Determine the force f in the case that the position $p(t) = e^{-t^2}$ and $M_1 = M_2 = \frac{1}{2}$ and $K_1 = K_2 = B = 1$.

Solution: Let p_1 be a signal representing the position of mass M_1. Suppose that the spring K_1 connecting masses M_1 and M_2 is a rest when the masses are distance d_1 apart, i.e., $p - p_1 = d_1$. The force applied by spring K_1 on mass M_2 is

$$f_1 = -K_1(p - p_1 - d_1) = -K_1(p - g)$$

where $g = p_1 + d_1$. The force applied by spring K_1 on mass M_1 is then $-f_1$. The force applied by the damper on M_1 is

$$f_d = -BDp_1 = -BD(g - d_1) = -BDg.$$

The total force applied to M_1 is $f + f_d - f_1$ and by Newton's law

$$M_1D^2p_1 = M_1D^2g = f + f_d - f_1 = f - BDg + K_1(p - g).$$

The force applied to M_2 by the spring K_2 is

$$f_2 = -K_2p$$

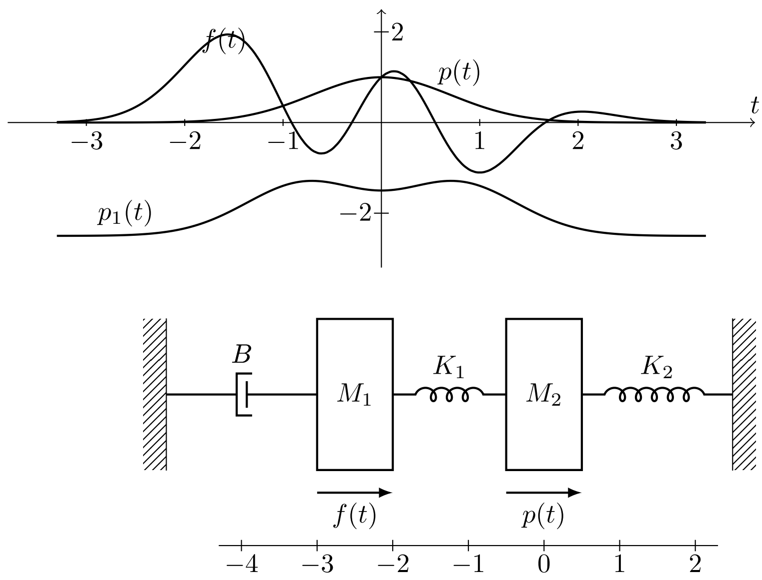


Figure 2.4: Motion of masses M_1 and M_2 when postition of masses M_1 is $p(t) = e^{-t^2}$.

because the spring is assumed to be at rest when $p = 0$. The total force applied to M_2 is $f_1 + f_2$ and by Newton's law

$$M_2 D^2 p = f_1 + f_2 = -K_1(p - g) - K_2 p.$$

Rearranging gives

$$K_1 g = (K_1 + K_2)p + M_2 D^2 p$$

and

$$-K_1(p - g) = M_2 D^2 p + K_2 p.$$

Now,

$$M_1 D^2 g + BDg + M_2 D^2 p + K_2 p = f$$

and so

$$K_1K_2p+B(K_1+K_2)Dp+(M_1K_1+M_1K_2+K_1M_2)D^2p+BM_2D^3p+M_1M_2D^4p = K_1 f.$$

In the case that $M_1 = K_1 = K_2 = B = 1$ and $M_2 = 2$ we have

$$p + 2Dp + 4D^2p + 2D^3p + 2D^4p = f.$$

and if $p(t) = e^{-t^2}$ we have

$$f(t) = (32t^4 - 16t^3 - 80t^2 + 20t + 17)e^{-t^2}, \qquad g(t) = (8t^2 - 2)e^{-t^2}$$

The solultion is animated in Figure 2.4 under the assumption that $d = 2.5$.

2.3. Consider the electromechanical system in Figure 2.6. A direct current motor is connected to a potentiometer in such a way that the voltage at the output of the potentiometer is equal to the angle of the motor θ. This voltage is fed back via a unity gain amplifier to the input terminal of the motor. An input voltage v is applied to the other terminal on the motor. Find the differential equation relating v and θ. What is the input voltage v if the motor angle satisfies $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$? Plot θ and v in this case when the motor coefficients satisfy $L = 0$, $R = \frac{3}{4}$, and $K_b = K_\tau = B = J = 1$.

Solution: The input voltage to the DC motor is $v - \theta$. From (2.4.1) of the lecture notes the relationship between the input voltage and motor angle is

$$v - \theta = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta$$

and so

$$v = \theta + \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta.$$

If $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$ then

$$D\theta(t) = \sqrt{\pi} e^{-t^2}, \qquad D^2\theta(t) = -2t\sqrt{\pi} e^{-t^2}$$

and so

$$v(t) = \frac{\pi\,(\mathrm{erf}\,(t) + 1)}{2} - 2\sqrt{\pi}\,t\,e^{-t^2} + 2\sqrt{\pi}\,e^{-t^2}$$

The signals v and θ are plotted in the figure below. Observe that as $t \to \infty$ both $\theta(t)$ and $v(t)$ converge to π.

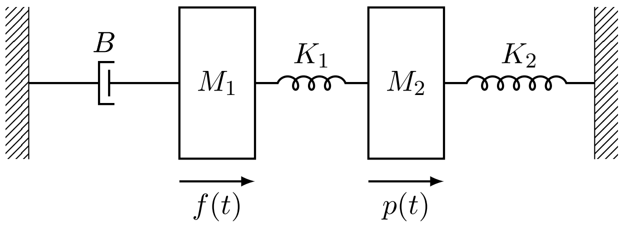


Figure 2.5: Two masses, a spring, and a damper connected between two walls for Exercise 2.2.

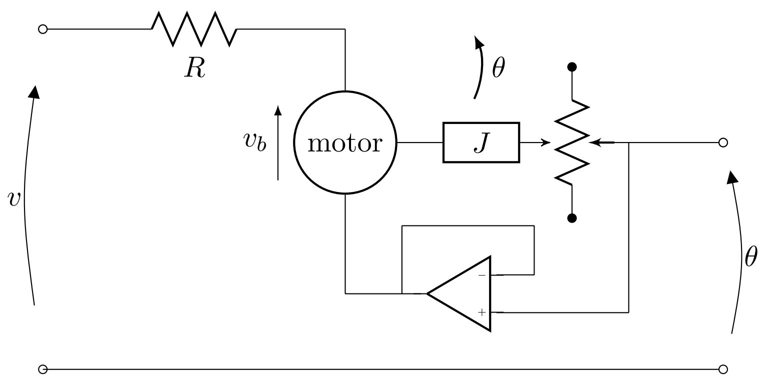


Figure 2.6: Diagram for a rotary direct current (DC) with potentiometer feedback for Exercise 2.3.

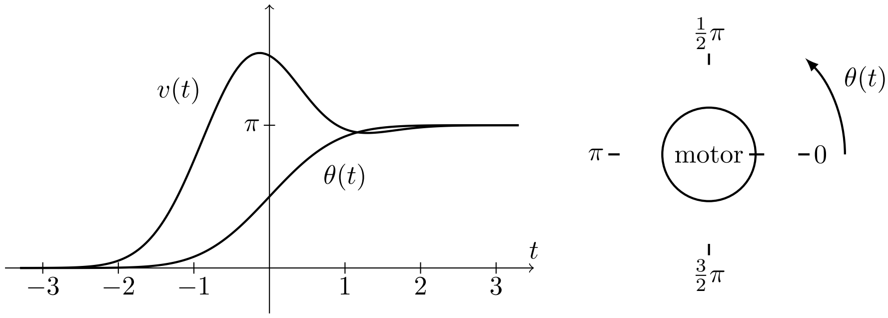


Figure 2.7: Voltage and corresponding angle for the dc motor with potentiometer in Figure 2.6 with constants $L = 0$, $R = \frac{3}{4}$, and$K_b = K_\tau = B = J = 1$.

Chapter 3

Linear time-invariant systems

Exercises

3.1. State whether each of the following systems are: causal, linear, shift-invariant, or stable. Plot the impulse and step response of the systems whenever they exist. In each case, assume the domain to be the set of locally integrable signals.

(a) $Hx(t) = 3x(t-1) - 2x(t+1)$
(b) $Hx(t) = \sin\big(2\pi x(t)\big)$
(c) $Hx(t) = t^2 x(t)$
(d) $Hx(t) = \int_{-1/2}^{1/2} \cos(\pi\tau) x(t+\tau) d\tau$

Solution:

(a) The system can be written as $H(x) = 3T_1(x) - 2T_{-1}(x)$ which is a linear combination of shifters. Since the shifter is linear and shift-invariant H will be also (Section 3.3 of the notes). Linearity can also be shown directly

$$\begin{aligned} H(ax+by) &= 3\big(ax(t-1) + by(t-1)\big) - 2\big(ax(t+1) + by(t+1)\big) \\ &= a\big(3x(t-1) - 2x(t+1)\big) + b\big(3y(t-1) - 2y(t+1)\big) \\ &= aHx + bHy. \end{aligned}$$

Shift-invariance can also be shown directly

$$\begin{aligned} T_\tau Hx(t) &= Hx(t-\tau) \\ &= 3x(t-1-\tau) - 2x(t+1-\tau) \\ &= 3T_\tau x(t-1) - 2T_\tau x(t+1) \\ &= HT_\tau x(t). \end{aligned}$$

The system is stable because for every input signal bounded less than $M > 0$, that is, for all input signals x such that $|x(t)| < M$ for all $t \in \mathbb{R}$, we can choose $K = 5M$ and

$$|Hx(t)| = |3x(t-1-\tau) - 2x(t+1-\tau)| \leq 3\,|x(t-1-\tau)| + 2\,|x(t+1-\tau)| < 5M = K,$$

i.e., the output signal is bounded less than K. The system is not causal, because it depends on the input signal x at time $t+1$, i.e., in the 'future'. The system is not regular because it is a linear combination of time shifters, and these are not regular, they don't formally have an impulse response (Section 3.1 of the notes). The system does have a step response equal to $H(u,t) = 3u(t-1) - 2u(t+1)$ that is plotted in the figures below.

(b) The system is causal, in fact, it is memoryless since it only depends on the input signal x at time t, i.e. the 'present' time. The system $Hx(t) = \sin\big(2\pi x(t)\big)$ is shift-invariant because

$$HT_\tau x(t) = \sin\big(2\pi T_\tau x(t)\big) = \sin\big(2\pi x(t-\tau)\big) = T_\tau Hx(t).$$

The system is not linear since $H(ax,t) = \sin\big(2\pi ax(t)\big) \neq a\sin\big(2\pi x(t)\big)$ in general. The system is stable, for every input signal x with absolute value bounded below M we have

$$|\sin(x(t))| = \left|\frac{e^{jx(t)} - e^{-jx(t)}}{2j}\right| \leq \frac{\left|e^{jx(t)}\right| + \left|e^{-jx(t)}\right|}{2} < e^M$$

and so, choosing $K = e^M$ we find that $|Hx(t)| < K$. Because the system is not linear, it is not regular. It does have a step response equal to $Hu(t) = \sin\big(2\pi u(t)\big) = 0$. Also acceptable is that is doesn't have a step response because this is a feature we developed for linear shift-invariant systems.

(c) The system is causal and also memoryless. The system $Hx(t) = t^2x(t)$ is linear because

$$H(ax+by) = t^2(ax+by) = at^2x + bt^2y = aHx + bHy.$$

The system is not shift-invariant because

$$T_\tau Hx(t) = (t-\tau)^2x(t-\tau) \neq HT_\tau x(t) = t^2x(t-\tau)$$

in general. The system is not regular because it is not shift-invariant. The system does not have an impulse response. It does have a step response equal to $Hu(t) = t^2u(t)$. This is plotted in the figures below. Also acceptable is that is doesn't have a step response because this is a feature we developed for linear shift-invariant systems. The system is not stable, for example the input step u is bounded below $M > 1$ but the output Hu is not bounded, it grows indefinitely as $t \to \infty$.

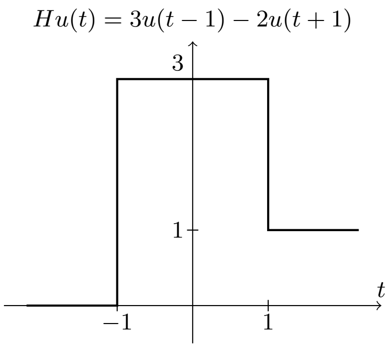
(d) Put $h(t) = \cos(\pi t)\Pi(t)$ where h is the rectangle function (see (1.1.2) of the lecture notes). Now

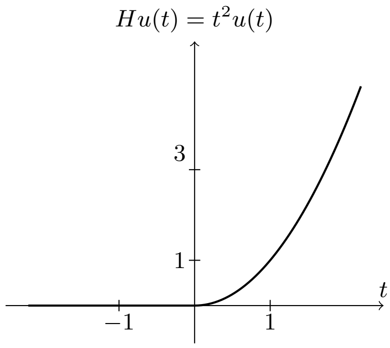
$$\begin{aligned}
Hx(t) &= \int_{-1/2}^{1/2} \cos(\pi\tau)x(t+\tau)d\tau \\
&= \int_{-1/2}^{1/2} \cos(-\pi\tau)x(t-\tau)d\tau \qquad\qquad \text{(change var } \tau = -\tau) \\
&= \int_{-\infty}^{\infty} \cos(-\pi\tau)\Pi(t)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= (h * x)(t).
\end{aligned}$$

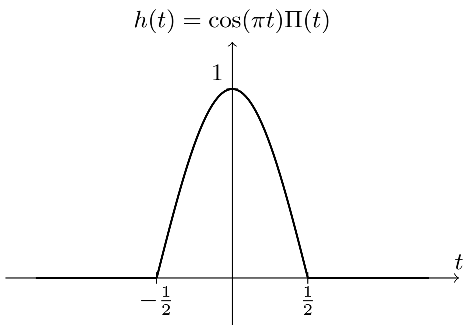
Thus, H is the regular system with impulse response $h(t) = \cos(\pi t)\Pi(t)$. A plot of h is given below. Since H is regular it is also linear and shift-invariant. The impulse response h is absolutely integrable with $\|h\|_1 = \frac{2}{\pi}$ and so H is stable. The system H is not causal because h is nonzero with some $t < 0$, specifically those

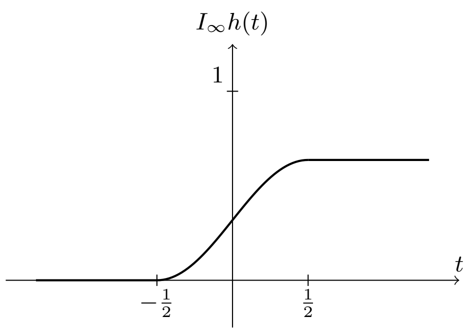
$t \in (-\frac{1}{2}, \frac{1}{2})$. The step response is given by applying the integrator system I_∞ to h, that is,

$$I_\infty h(t) = \int_{-\infty}^{t} h(\tau)d\tau = \begin{cases} 0 & t \leq -\frac{1}{2} \\ \int_{-1/2}^{t} \cos(\pi\tau)d\tau = \big(\sin(\pi t) + 1\big)/\pi & -\frac{1}{2} < t \leq \frac{1}{2} \\ \frac{2}{\pi} & t > \frac{1}{2}. \end{cases}$$









3.2. Show that the system $Hx(t) = \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau$ is linear shift-invariant and regular. Find and sketch the impulse response and the step response. **Solution:** The easy way is to spot the impulse response directly.

Observe that

$$\begin{aligned}
H(x)(t) &= \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau \\
&= \int_{-\infty}^{\infty} \Pi(\tau/2)\sin(\pi\tau)x(t+\tau)d\tau \\
&= -\int_{\infty}^{-\infty} \Pi(-\tau/2)\sin(-\pi\tau)x(t-\tau)d\tau \qquad \text{(ch. var. } \tau \to -\tau) \\
&= -\int_{-\infty}^{\infty} \Pi(\tau/2)\sin(\pi\tau)x(t-\tau)d\tau \\
&= (h * x)(t),
\end{aligned}$$

where we put $h(t) = -\Pi(t/2)\sin(\pi t)$. It follows that h is the impulse response of H. Since h has an impulse resposne it is regular, and since it is regular its also linear and time invariant.

The hard way is to first show linear, then show time invariance, and then find this impulse response as the limit

$$h = \lim_{\gamma\to\infty} Hp_\gamma.$$

where the function

$$p_\gamma(t) = \begin{cases} \gamma, & 0 < t \le \frac{1}{\gamma} \\ 0, & \text{otherwise,} \end{cases}$$

is introduced in Section 3.1. We have

$$\begin{aligned}
H(ax+by) &= \int_{-1}^{1} \sin(\pi\tau)\big(ax(t+\tau)+by(t+\tau)\big)d\tau \\
&= a\int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau + b\int_{-1}^{1} \sin(\pi\tau)y(t+\tau)\big)d\tau \\
&= aH(x) + bH(y),
\end{aligned}$$

and so, H is linear. We also have

$$\begin{aligned}
H\big(T_k(x)\big) &= \int_{-1}^{1} \sin(\pi\tau)T_k(x)(t+\tau)d\tau \\
&= \int_{-1}^{1} \sin(\pi\tau)x(t+\tau-k)d\tau \\
&= T_k\left(\int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau\right) \\
&= T_k\big(H(x)\big),
\end{aligned}$$

and so, H is time invariant. Now, if H is regular then its impulse response is $h = \lim_{\gamma\to\infty} H(p_\gamma)$. Let h_γ be the signal

$$h_\gamma(t) = \int_{-1}^{1} \sin(\pi\tau)p_\gamma(t+\tau)d\tau.$$

The impulse response exists if h_γ converges for each fixed t as $\gamma \to \infty$. Now, $p_\gamma(t+\tau) = \gamma$ for $t+\tau \in [0, \frac{1}{\gamma})$, i.e. $\tau \in [-t, \frac{1}{\gamma} - t)$, and zero otherwise. The integral ranges from -1 to 1 so we are also interested in those $\tau \in [-1, 1]$. When $t > \frac{1}{\gamma} + 1$ or $t < -1$ the intervals $[-1, 1]$ and $[-t, \frac{1}{\gamma} - t)$ are disjoint and we obtain $h(t) = 0$.

Otherwise, when $[-t, \frac{1}{\gamma} - t) \subset [-1, 1]$, i.e, $-t > -1$ and $\frac{1}{\gamma} - t < 1$ we obtain

$$\begin{aligned} h_\gamma(t) &= \int_{-1}^{1} \sin(\pi\tau) p_\gamma(t+\tau) d\tau \\ &= \gamma \int_{-t}^{1/\gamma - t} \sin(\pi\tau) d\tau \\ &= -\frac{\gamma}{\pi}\big(\cos\big(\pi(1/\gamma - t)\big) - \cos(-\pi t)\big) \\ &= -\frac{\gamma}{\pi}\big(\cos\big(\pi(t - \tfrac{1}{\gamma})\big) - \cos(\pi t)\big). \end{aligned}$$

Put $\Delta = -\frac{1}{\gamma}$ and

$$h_\gamma(t) = \frac{1}{\pi} \frac{\cos\big(\pi(t+\delta)\big) - \cos(\pi t)}{\delta}.$$

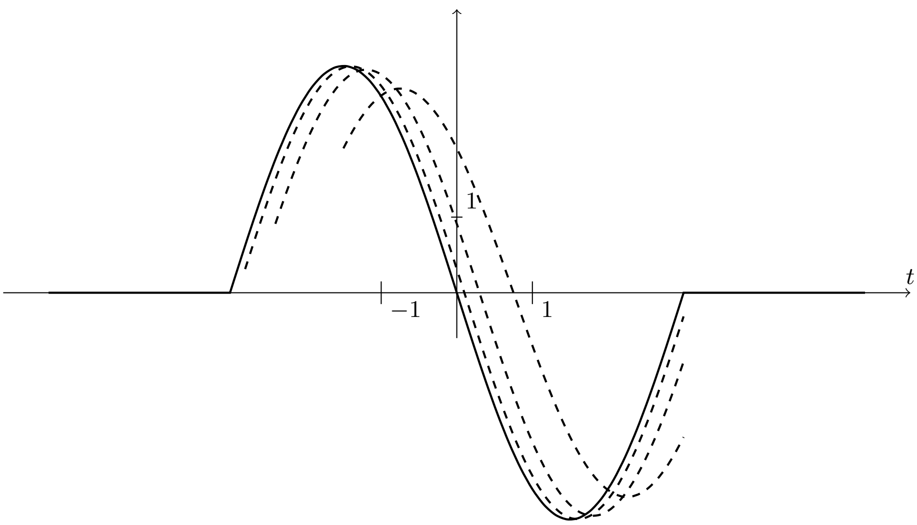
Recognising the limit as $\gamma \to \infty$, or equivalently as $\Delta \to 0$ as

$$\lim_{\delta \to 0} \frac{\cos\big(\pi(t+\delta)\big) - \cos(\pi t)}{\delta} = \frac{d}{dt}\cos(\pi t)$$

we immediately have

$$\lim_{\gamma \to \infty} h_\gamma(t) = h(t) = \frac{1}{\pi}\frac{d}{dt}\cos(\pi t) = -\sin(\pi t).$$

on the interval $t \in [\frac{1}{\gamma} - 1, 1)$. It remains to show what happens on the interval $[-1, \frac{1}{\gamma} - 1)$ that shrinks as $\gamma \to \infty$.



The step response can be found directly by inputing the step function u to the system. That is

$$Hu(t) = \int_{-1}^{1} \sin(\pi\tau) u(t+\tau) d\tau.$$

To find an explicit expression for this integral 3 cases must be considered separately. Observe that $u(t+\tau)$ is nozero only when $\tau > -t$. If $t < -1$ then $u(t+\tau) = 0$ for all $\tau \in [-1, 1]$ and so

$$Hu(t) = \int_{-1}^{1} \sin(\pi\tau) u(t+\tau) d\tau = 0 \qquad t < -1.$$

If $t > 1$ then $u(t+\tau) = 1$ for all $\tau \in [-1, 1]$ and so

$$Hu(t) = \int_{-1}^{1} \sin(\pi\tau) d\tau = -\frac{\cos(\pi\tau)}{\pi}\Big|_{-1}^{1} = \frac{-\cos(\pi) + \cos(-\pi)}{\pi} = 0 \qquad t > 1.$$

Finally, if $-1 \leq t \leq 1$ then $u(t+\tau)$ is 1 for $\tau \in [-t, 1]$ and 0 for $\tau \in [-1, -t)$ and so

$$\begin{aligned} Hu(t) &= \int_{-t}^{1} \sin(\pi\tau) d\tau \\ &= -\frac{\cos(\pi\tau)}{\pi}\Big|_{-t}^{1} \\ &= \frac{-\cos(\pi) + \cos(-\pi t)}{\pi} = \frac{\cos(\pi t) + 1}{\pi} \qquad 1 \leq t \leq 1. \end{aligned}$$

An alternative way to find the step response is to apply the integrator system I_∞ to the impulse response $h(t) = -\Pi(t/2)\sin(\pi t)$ we derived earlier. We have

$$Hu(t) = I_\infty h(t) = -\int_{-\infty}^{t} \Pi(\tau/2)\sin(\pi\tau) d\tau.$$

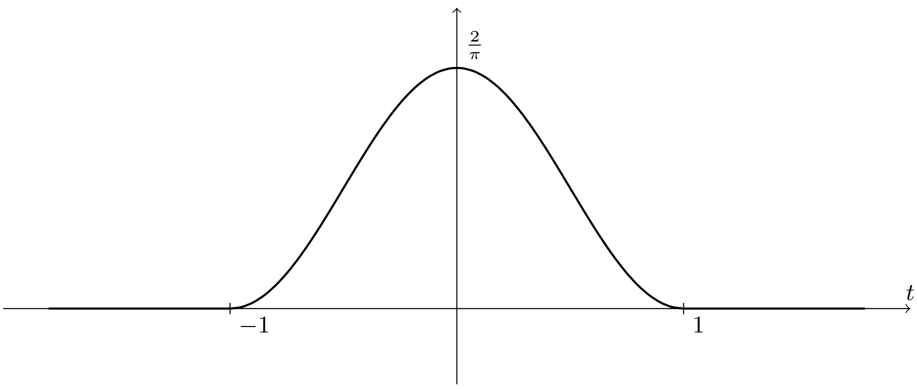
Again the integral needs to be split into cases. When $t < -1$ the $\Pi(\tau/2)$ occuring inside the integral is always zero and so $H(u, t) = 0$ for $t < -1$. When $t > 1$

$$Hu(t) = -\int_{-1}^{1} \sin(\pi\tau) d\tau = 0.$$

Finally, when $-1 \leq t \leq 1$ we have

$$Hu(t) = -\int_{-1}^{t} \sin(\pi\tau) d\tau = \frac{\cos(\pi\tau)}{\pi}\Big|_{-1}^{t} = \frac{\cos(\pi t) + 1}{\pi}.$$

Observe that this is the same as previously. The step response is plotted below.



3.3. Let h be a locally integrable signal. Show that the set $\operatorname{dom} h$ defined in Section 3.1 on page 33 is a linear shift-invariant space. **Solution:** The set $\operatorname{dom} h$ contains those signals x for which

$$\int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Suppose that $x, y \in \operatorname{dom} h$ and $a, b \in \mathbb{C}$. Then

$$\begin{aligned} &\int_{-\infty}^{\infty} |h(\tau)(ax(t-\tau) + by(t-\tau)|\, d\tau \\ &\qquad \leq |a|\int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau + |b|\int_{-\infty}^{\infty} |h(\tau)y(t-\tau)|\, d\tau. \end{aligned}$$

Both integrals on the right hand side are finite for all $t \in \mathbb{R}$ becase x and y are in $\operatorname{dom} h$. Thus the right hand side is finite for all $t \in \mathbb{R}$ and so the linear combination $ax + by \in \operatorname{dom} h$. It follows that $\operatorname{dom} h$ is a linear space.

If $x \in \operatorname{dom} h$ then

$$\int_{-\infty}^{\infty} |h(\tau) T_k x(t-\tau)| \, d\tau = \int_{-\infty}^{\infty} |h(\tau) x(t-\tau-k)| \, d\tau < \infty$$

for all $t \in \mathbb{R}$ and so the shifted signal $T_k x \in \operatorname{dom} h$. It follows that $\operatorname{dom} h$ is a shift-invariant space.

3.4. Show that $\operatorname{dom} u$ where u is the step function is the subset of locally integrable signals such that $\int_{-\infty}^{0} |x(t)| \, dt < \infty$. **Solution:** By definition $\operatorname{dom} u$ is the set of signals x such that

$$\int_{-\infty}^{\infty} |u(\tau) x(t-\tau)| \, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Denote by B the subset of locally integrable signals such that $\int_{-\infty}^{0} |x(t)| \, dt < \infty$. We first show that $\operatorname{dom} u$ is a subset of B, that is $\operatorname{dom} u \subseteq B$. We do so by contraposition, that is, we show that if $x \notin B$ then $x \notin \operatorname{dom} u$. Suppose that x is not locally integrable, that is, suppose there exists $a, b \in \mathbb{R}$ such that $\int_a^b |x(t)| \, d\tau$ is not finite. Then $x \notin B$. Now

$$\int_{-\infty}^{\infty} |u(\tau) x(t-\tau)| \, d\tau = \int_{-\infty}^{\infty} |u(t-k) x(k)| \, dk = \int_{-\infty}^{t} |x(k)| \, dk$$

the second equation following from the change of variable $k = t - \tau$. Choosing $k > b$ we have

$$\int_{-\infty}^{\infty} |u(\tau) x(t-\tau)| \, d\tau = \int_{-\infty}^{t} |x(\tau)| \, d\tau \geq \int_{a}^{b} |x(\tau)| \, d\tau$$

which, by assumption, is not finite, and so $x \notin \operatorname{dom} u$.

We now show that $B \subseteq \operatorname{dom} u$. Suppose that $x \in \operatorname{dom} u$, that is, suppose that

$$\int_{-\infty}^{\infty} |u(\tau) x(t-\tau)| \, d\tau < \infty$$

for all t. Then

$$\int_{-\infty}^{\infty} |u(\tau) x(t-\tau)| \, d\tau = \int_{-\infty}^{t} |x(\tau)| \, d\tau = \int_{-\infty}^{a} |x(\tau)| \, d\tau + \int_{a}^{t} |x(\tau)| \, d\tau$$

for all $a, t \in \mathbb{R}$ and so, the two integrals on the right are finite for all $a, t \in \mathbb{R}$. In particular

$$\int_{a}^{t} |x(\tau)| \, d\tau < \infty$$

for all $a, t \in \mathbb{R}$ and so x is locally integrabls and putting $a = 0$ we have that

$$\int_{-\infty}^{0} |x(\tau)| \, d\tau < \infty.$$

It follows that $x \in B$. We have now show that $\operatorname{dom} u \subseteq B$ and that $B \subseteq \operatorname{dom} u$ and so it must be that $B = \operatorname{dom} u$.

3.5. Show that a regular system is stable if and only if its impulse response is absolutely integrable. **Solution:** Let H be a regular system and h its impulse response. By default the domain of H is assumed to be $\operatorname{dom} h$, that is $H \in \operatorname{dom} h \to \mathbb{C}$. If h is absolutely integrable then for all signals $x \in \operatorname{dom} h$ such that $|x(t)| < M$ for all t,

$$\begin{aligned}
|Hx(t)| &= |(h * x)(t)| \\
&= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau \\
&< \int_{-\infty}^{\infty} M|h(\tau)|d\tau \\
&= M\|h\|_1
\end{aligned}$$

for all t, and so $Hx(t)$ is bounded.

On the other hand if h is not absolutely integrable then consider the bounded signal

$$s(t) = \begin{cases} 1 & h(-t) > 0, \\ -1 & h(-t) \leq 0. \end{cases}$$

Observe that s is not in the domain $\operatorname{dom} h$ because

$$\int_{-\infty}^{\infty} |h(\tau)s(-\tau)|\, d\tau = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty$$

However, for all $\kappa > 0$ the signal

$$r_\kappa(t) = \Pi\big(\tfrac{t}{2\kappa}\big)s(t) = \begin{cases} s(t) & |t| < \kappa \\ 0 & \text{otherwise} \end{cases}$$

is in $\operatorname{dom} h$ since

$$\int_{-\infty}^{\infty} |h(\tau)r_\kappa(-\tau)|\, d\tau = \int_{-\kappa}^{\kappa} |h(\tau)|\, d\tau < \infty$$

because h is locally integrable (the impulse response is always locally integrable by assumption. See Section 3.1). Put $M > 1$ and suppose that H was stable. Then there exists $K > 0$ such that $|Hx(t)| < K$ for all $t \in \mathbb{R}$ and all $x \in \operatorname{dom} h$ bounded less that M. Observe that r_κ is bounded less than M, that is, $|r_\kappa(t)| \leq 1 < M$ for all $\kappa \in \mathbb{R}$ and all $t \in \mathbb{R}$. The response of H to r_κ at time zero is

$$Hr_\kappa(0) = \int_{-\infty}^{\infty} h(\tau)r_\kappa(-\tau)d\tau = \int_{-\kappa}^{\kappa} |h(\tau)|\, d\tau$$

and because h is not absolutely integrable the integral on the right diverges as κ get large. Thus, we can choose κ large enough that

$$|Hr_\kappa(0)| = Hr_\kappa(0) = \int_{-\kappa}^{\kappa} |h(\tau)|\, d\tau > K$$

violating our assumption that H was stable. Thus, H is not stable.

3.6. Define signals $x(t) = u(t)$, $y(t) = u(-t)$, and $z(t) = \Pi(t) - \Pi(t-1)$ where u is the step function and Π is the rectangular pulse. Plot x, y, and z and show that the associative property of convolution does not hold for these signals. That is, show that $x * (y * z) \neq (x * y) * z$.

3.7. Show that $\sum_{\ell=1}^{L} e^{\beta\ell} = \frac{e^{\beta(L+1)}-e^{\beta}}{e^{\beta}-1}$ (Hint: sum a geometric progression).
Solution: Put $r = e^{\beta}$ and put

$$S_L = \sum_{\ell=1}^{L} e^{\beta\ell} = \sum_{\ell=1}^{L} r^{\ell}.$$

This is the sum of the first L terms of a geometric progression. We have

$$rS_L - S_L = r^{L+1} - r$$

and so

$$S_L = \frac{r^{L+1} - r}{r - 1} = \frac{e^{\beta(L+1)} - e^{\beta}}{e^{\beta} - 1}$$

as required.

3.8. Show that

$$\frac{2j}{L}\sum_{\ell=1}^{L} \sin(\gamma\ell - \theta)e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where $\alpha = e^{-j\theta}$ and $C = e^{-j\gamma(L+1)}\frac{\sin(\gamma L)}{L\sin(\gamma)}$. (Hint: solve Exercise 3.7 first and then use the formula $2j\sin(x) = e^{jx} - e^{-jx}$). **Solution:** We have

$$2j\sin(\gamma\ell - \theta) = e^{j(\gamma\ell-\theta)} - e^{-j(\gamma\ell-\theta)}$$

and so the sum becomes

$$\begin{aligned}\frac{1}{L}\sum_{\ell=1}^{L}(e^{j(\gamma\ell-\theta)} - e^{-j(\gamma\ell-\theta)})e^{-j\gamma\ell} &= \frac{1}{L}\sum_{\ell=1}^{L} e^{-j\theta} - \frac{1}{L}\sum_{\ell=1}^{L} e^{-2j\gamma}e^{j\theta} \\ &= \alpha - \frac{\alpha^*}{L}\sum_{\ell=1}^{L} e^{-2j\gamma}.\end{aligned}$$

The sum is a geometric progression and, using the answer to Exercise 3.7, we have

$$\sum_{\ell=1}^{L} e^{-2j\gamma} = \frac{e^{-2j\gamma(L+1)} - e^{-2j\gamma})}{e^{-2j\gamma} - 1}.$$

The denominator satisfies

$$e^{-2j\gamma} - 1 = e^{-j\gamma}(e^{-j\gamma} - e^{j\gamma}) = -2je^{-j\gamma}\sin(\gamma).$$

The numerator satisfies

$$\begin{aligned}e^{-2j\gamma(L+1)} - e^{-2j\gamma} &= e^{-2j\gamma}(e^{-2j\gamma L} - 1) \\ &= e^{-2j\gamma}e^{-j\gamma L}(e^{-j\gamma L} - e^{j\gamma L}) \\ &= -2je^{-j\gamma(L+2)}\sin(\gamma L).\end{aligned}$$

Thus

$$\sum_{\ell=1}^{L} e^{-2j\gamma} = \frac{-2je^{-j\gamma(L+2)}\sin(\gamma L)}{-2je^{-j\gamma}\sin(\gamma)} = \frac{e^{-j\gamma(L+1)}\sin(\gamma L)}{\sin(\gamma)} = LC$$

where C is defined in the question statement. Now

$$\frac{2j}{L}\sum_{\ell=1}^{L}\sin(\gamma\ell - \theta)e^{-j\gamma\ell} = \alpha - \frac{\alpha^*}{L}LC = \alpha - \alpha^* C$$

as required.

$*$3.9. Show that the convolution of two absolutely integrable signals is absolutely integrable. **Solution:** Let x and y be absolutely integrable. We want to show that the convolution

$$(x * y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

is absolutely integrable. Write

$$\begin{aligned}
\|x * y\|_1 &= \int_{-\infty}^{\infty} |(x * y)(t)|\, dt \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \right| dt \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau)y(t-\tau)|\, d\tau dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau)|\, |y(t-\tau)|\, dt d\tau \qquad \text{(change order of integration, Tonelli's theorem)} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y(t-\tau)|\, dt\, |x(\tau)|\, d\tau \\
&= \|y\|_1 \int_{-\infty}^{\infty} |x(\tau)|\, d\tau \\
&= \|y\|_1 \|x\|_1
\end{aligned}$$

which is finite by our assumption that x and y are absolutely integrable. This result also follows as a special case of Young's Theorem [Rudin, 1986].

Chapter 4

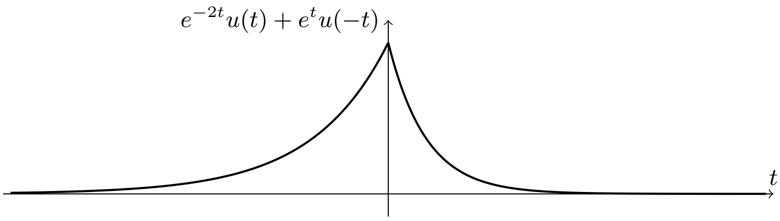
The Laplace transform

Exercises

4.1. Sketch the signal

$$x(t) = e^{-2t}u(t) + e^{t}u(-t)$$

where $u(t)$ is the step function. Find the Laplace transform of $x(t)$ and the corresponding region of convergence. Sketch the region of convergence on the complex plane. **Solution:**



The Laplace transform of $e^{-2t}u(t)$ is

$$\begin{aligned}
\mathcal{L}(e^{-2t}u(t), s) &= \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt \\
&= \int_{0}^{\infty} e^{-(s+2)t}dt \\
&= -\frac{e^{-(s+2)t}}{s+2}|_0^{\infty} \\
&= \frac{1}{s+2}, \qquad \mathrm{Re}(s) > -2
\end{aligned}$$

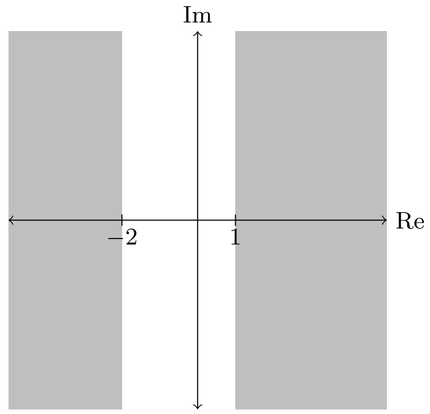
and the Laplace transform of $e^{t}u(-t)$ is

$$\begin{aligned}
\mathcal{L}(e^{t}u(-t), s) &= \int_{-\infty}^{0} e^{-(s-1)t}dt \\
&= -\frac{e^{-(s-1)t}}{s-1}|_{-\infty}^{0} \\
&= -\frac{1}{s-1}, \qquad \mathrm{Re}(s) < 1.
\end{aligned}$$

Thus, the Laplace transform of $e^{-2t}u(t) + e^{t}u(-t)$ is

$$\mathcal{L}(e^{-2t}u(t) + e^{t}u(-t), s) = \frac{1}{s+2} - \frac{1}{s-1}, \qquad -2 < \operatorname{Re}(s) < 1$$

and the region of convergence is the subset of the complex plane satisfying $-2 < \operatorname{Re}(s) < 1$. The unshaded region in the plot below depicts the ROC.



4.2. Find the Laplace transform of the signal $t^n u(t)$ where $n \geq 0$ is an integer. **Solution:** We have

$$\mathcal{L}\big(t^n u(t)\big) = \int_{-\infty}^{\infty} t^n u(t) e^{-st} dt = \int_{0}^{\infty} t^n e^{-st} dt.$$

Integration by parts gives the indefinite integral

$$\int t^n e^{-st} dt = \frac{t^n}{s} e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st} dt.$$

So, when $\operatorname{Re}(s) > 0$,

$$\begin{aligned}\mathcal{L}\big(t^n u(t)\big) &= \lim_{t \to 0} \frac{t^n}{s} e^{-st} - \lim_{t \to \infty} \frac{t^n}{s} e^{-st} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt \\ &= \frac{n}{s} \mathcal{L}\big(t^{n-1} u(t)\big),\end{aligned}$$

since both limits converge to zero. Unravelling the above recursive equation gives

$$\mathcal{L}\big(t^n u(t)\big) = \frac{n}{s} \times \frac{n-1}{s} \times \cdots \times \frac{1}{s} \times \mathcal{L}\big(u(t)\big) = \frac{n!}{s^{n+1}}, \qquad \operatorname{Re}(s) > 0,$$

since $\mathcal{L}\big(u(t)\big) = \frac{1}{s}$ when $\operatorname{Re}(s) > 0$.

4.3. Let $n \geq 0$ be an integer. Show that the Laplace transform of the signal $(-t)^n u(-t)$ is the same as the Laplace transform of the signal $t^n u(t)$, but with a different region of convergence. **Solution:** We have

$$\begin{aligned}\mathcal{L}\big((-t)^n u(-t)\big) &= \int_{-\infty}^{\infty} (-t)^n u(-t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} t^n u(t) e^{st} dt \qquad \text{(change variable t = -t)} \\ &= \mathcal{L}(t^n u(t), -s) \qquad \operatorname{Re}(s) < 0 \\ &= \frac{n!}{s^{n+1}} \qquad \operatorname{Re}(s) < 0.\end{aligned}$$

4.4. Show that equation (4.3.5) on page 57 holds when the system H the time shifter T_τ. **Solution:** Put $y = T_\tau x = x(t-\tau)$. Taking Laplace transforms

$$\begin{aligned}
\mathcal{L}y &= \mathcal{L}T_\tau x \\
&= \int_{-\infty}^{\infty} T_\tau x(t) e^{-st} dt \\
&= \int_{-\infty}^{\infty} x(t-\tau) e^{-st} dt \\
&= \int_{-\infty}^{\infty} x(\kappa) e^{-s(\kappa+\tau)} d\kappa \qquad \text{(ch. vars. } \kappa = t-\tau) \\
&= e^{-s\tau} \int_{-\infty}^{\infty} x(\kappa) e^{-s\kappa} d\kappa \\
&= e^{-s\tau} \mathcal{L}x \qquad s \in \mathrm{roc}(x) \\
&= \lambda T_\tau \mathcal{L}x \qquad s \in \mathrm{roc}(x)
\end{aligned}$$

as required. Observe that the region of convergence of $\mathcal{L}y$ is the same as that of $\mathcal{L}x$.

4.5. Show that equation (4.3.5) on page 57 holds when the system H is the differentiator under the added assumption that

$$\lim_{t\to\infty} x(t)e^{-st} = \lim_{t\to-\infty} x(t)e^{-st} = 0 \qquad \text{when } s \in \mathrm{roc}(x).$$

Solution:

Put $y = Dx$. Taking Laplace transforms

$$\mathcal{L}y = \mathcal{L}Dx = \int_{-\infty}^{\infty} Dx(t) e^{-st} dt.$$

Integrating by parts

$$\mathcal{L}y = \big[x(t)e^{-st}\big]_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st} dt = \big[x(t)e^{-st}\big]_{-\infty}^{\infty} + s\mathcal{L}x.$$

and, by assumption,

$$\big[x(t)e^{-st}\big]_{-\infty}^{\infty} = \lim_{t\to\infty} x(t)e^{-st} - \lim_{t\to-\infty} x(t)e^{-st} = 0$$

whenever s is in the region of convergence of x. In this case $\mathcal{L}y = s\mathcal{L}x$ as required. The result follows for the kth differentiator D^k under the assumption that

$$\lim_{t\to\infty} D^c x(t)e^{-st} = 0 \qquad \text{and} \qquad \lim_{t\to-\infty} D^c x(t)e^{-st} = 0$$

for all $c = 1, 2, \dots, k-1$ because

$$\mathcal{L}D^k x = \mathcal{L}DD^{k-1}x = s\mathcal{L}D^{k-1}x$$

and unravelling this recursion gives

$$\mathcal{L}D^k y = \underbrace{s \times s \times \cdots \times s}_{k-1 \text{ times}} \times \mathcal{L}Dy = s^k \mathcal{L}y = \lambda D^k \mathcal{L}y.$$

4.6. Let x be the signal with Laplace transform

$$\mathcal{L}(x,s) = \frac{1}{(s-1)^3} \qquad \mathrm{Re}(s) > 1.$$

Define the signal y by

$$y(t) = e^t x(2t+1).$$

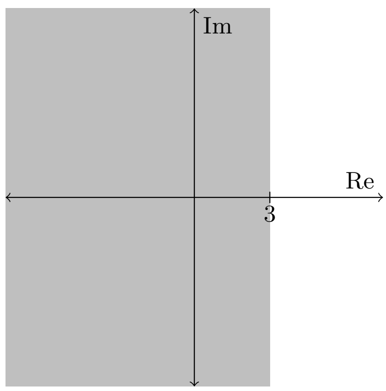
Find the Laplace transform and region of convergence of y. Sketch the region of convergence of y. **Solution:** The easiest approach is the write

$$\begin{aligned}
\mathcal{L}(y,s) &= \int_{-\infty}^{\infty} y(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} e^t x(2t+1)e^{-st}dt \\
&= \int_{-\infty}^{\infty} x(2t+1)e^{-(s-1)t}dt \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} x(k)e^{-(s-1)(k-1)/2}dk \qquad \text{(c.v. } k = 2t+1) \\
&= \tfrac{1}{2}e^{(s-1)/2}\int_{-\infty}^{\infty} x(k)e^{-k(s-1)/2}dk
\end{aligned}$$

The integral is the Laplace transform of x evaluated as $\frac{s-1}{2}$ and so

$$\mathcal{L}(y,s) = \tfrac{1}{2}e^{(s-1)/2}\mathcal{L}\big(x, \tfrac{s-1}{2}\big) = \frac{4e^{(s-1)/2}}{(s-3)^3} \qquad \mathrm{Re}\left(\tfrac{s-1}{2}\right) > 1.$$

The region of convergence of y is those complex numbers with real part greater than 3. A plot of the region of convergence is below.



A direct approach is to apply the inverse Laplace transform (by formula (4.2.3)) to find

$$x(t) = \tfrac{1}{2}t^2 e^t u(t)$$

where $u(t)$ is the step function. Now

$$y(t) = e^t x(2t+1) = \tfrac{1}{2}(2t+1)^2 e^{3t+1}u(2t+1).$$

One can now apply the Laplace transform formula to this expression for y. After a lengthy integration by parts, the same answer is obtained.

4.7. What is the transfer function of the integrator system I_∞? What is the domain of this transfer function? **Solution:** We can do this in two ways. First by obtaining the transfer function directly and second by using the fact that the impulse response of I_∞ is the step signal u (Section 3.1).

Recall from Section 1.4 that by default the domain of I_∞ is the set of locally integrable signals x for which the integral $\int_{-\infty}^{0} |x(t)|\, dt$. This set of precisley $\operatorname{dom} u$ (see also Exercise 3.4). Observe first that $e^{st} \in \operatorname{dom} u$ if and only if the real part of s is nonegative. Thus, the domain of the transfer function λI_∞ is the set of complex numbers s with $\operatorname{Re} s > 0$. The response of I_∞ to e^{st} is

$$I_\infty(e^{st}) = \int_{-\infty}^{t} e^{s\tau} d\tau = \frac{e^{st}}{s} - \lim_{t\to-\infty} \frac{e^{st}}{s}$$

and the limit exists only when $\operatorname{Re} s > 0$ and in this case it is zero. So

$$I_\infty(e^{st}) = \frac{1}{s} e^{st} \qquad \operatorname{Re} s > 0$$

and $\lambda I_\infty(s) = \frac{1}{s}$.

The second approach is to use that $\lambda H = \mathcal{L}u$. We have, from Section 4.1, that that the Laplace transform of the signal $e^{\alpha t}u(t)$ takes the form

$$\mathcal{L}(e^{\alpha t}u(t)) = \frac{1}{s-\alpha} \qquad \operatorname{Re} s > \operatorname{Re} \alpha.$$

Setting $\alpha = 0$ we find that

$$\mathcal{L}u(s) = \lambda I_\infty(s) = \frac{1}{s} \qquad \operatorname{Re} s > 0$$

as before.

4.8. By partial fractions, or otherwise, assert that

$$\frac{as}{s+b} = a - \frac{ab}{s+b}$$

Solution: Adding and subtracting ab from the numerator

$$\frac{as + ab - ab}{s+b} = \frac{a(s+b) - ab}{s+b} = \frac{a(s+b)}{s+b} - \frac{ab}{s+b} = a - \frac{ab}{s+b}$$

4.9. By partial fractions, or otherwise, assert that

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}$$

Solution: Hypothesise the solution

$$\frac{s+c}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}.$$

Multiplying both sides by $(s+a)(s+b)$,

$$s + c = A(s+b) + B(s+a).$$

Putting $s = -a$ gives $c - a = A(b-a)$, and pitting $s = -b$ gives $c - b = B(a-b)$, and so,

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}.$$

∗4.10. By partial fractions, or otherwise, assert that

$$\frac{1}{s(s-a)(s-b)(s-b^*)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_2^*}{s-b^*}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$ and $\mathrm{Im}(b) \neq 0$ and

$$A_0 = -\frac{1}{a|b|^2}, \qquad A_1 = \frac{1}{a|a-b|^2}, \qquad A_2 = \frac{1}{b(b-a)(b-b^*)}.$$

You might wish to check your solution using a symbolic programming language (for example Sage, Mathematica, or Maple). **Solution:** The mathemtica command

```
Apart[1/s/(s - a)/(s - b)/(s - c), s]
```

or the maxima command

```
partfrac(1/s/(s - a)/(s - b)/(s - c), s)
```

returns the equation

$$\frac{1}{s(s-a)(s-b)(s-c)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_3}{s-c}$$

where

$$A_0 = -\frac{1}{abc}, \qquad A_1 = \frac{1}{a(a-b)(a-c)},$$
$$A_2 = \frac{1}{b(b-a)(b-c)}, \qquad A_3 = \frac{1}{c(c-a)(c-b)}.$$

Setting $c = b^*$ gives

$$A_0 = -\frac{1}{a|b|^2}, \qquad A_1 = \frac{1}{a|a-b|^2},$$
$$A_2 = \frac{1}{b(b-a)(b-b^*)}, \qquad A_3 = \frac{1}{b^*(b^*-a)(b^*-b)} = A_2^*$$

as required.

4.11. Let y be a signal with Laplace transform taking the form

$$\mathcal{L}y(s) = \frac{2s+1}{s^2+s-2}$$

By partial fractions, or otherwise, find all possible signals y with this Laplace transform and the corresponding region of convergence. **Solution:** Factorise the polynomial on the denominator

$$\frac{2s+1}{(s+2)(s-1)}.$$

Adding and subtracting $s-1$ on the numerator

$$\frac{2s+1+(s-1)-(s-1)}{(s+2)(s-1)} = \frac{s-1}{(s-1)(s+2)} + \frac{s+2}{(s-1)(s+2)}$$
$$= \frac{1}{s+2} + \frac{1}{s-1}.$$

There are two time domain signals with Laplace transform $\frac{1}{s+2}$,

$$e^{-2t}u(t),\ \mathrm{Re}(s) > -2 \qquad \text{and} \qquad -e^{-2t}u(-t),\ \mathrm{Re}(s) < -2,$$

and two time domain signals with Laplace transform $-\frac{1}{s-1}$,

$$e^{t}u(t),\ \mathrm{Re}(s) > 1 \qquad \text{and} \qquad -e^{t}u(-t),\ \mathrm{Re}(s) < 1.$$

There are three possible signals with nonempty regions of convergence

$$y(t) = e^{-2t}u(t) - e^{t}u(-t) \qquad -2 < \mathrm{Re}(s) < 1,$$
$$y(t) = e^{-2t}u(t) + e^{t}u(t) \qquad 1 < \mathrm{Re}(s),$$
$$y(t) = -e^{-2t}u(-t) - e^{t}u(-t) \qquad \mathrm{Re}(s) < -2.$$

4.12. Let x be a signal. Show that the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ satisfies equation (4.2.4) on page 55. **Solution:** First consider when $\alpha = -1$ so that $x(-t)$ is the reflection of the signal x in time (see Section 1.4). We have

$$\begin{aligned}
\mathcal{L}\big(x(-t), s\big) &= \int_{-\infty}^{\infty} x(-t)e^{-st}dt \\
&= -\int_{\infty}^{-\infty} x(\tau)e^{s\tau}d\tau && \text{(change variable } \tau = -t) \\
&= \int_{-\infty}^{\infty} x(\tau)e^{s\tau}d\tau = \mathcal{L}(x, -s) \qquad \mathrm{Re}(-s) \in R.
\end{aligned}$$

This special case is called the **time reversal property**. Now, when $\alpha > 0$,

$$\begin{aligned}
\mathcal{L}\big(x(\alpha t), s\big) &= \int_{-\infty}^{\infty} x(\alpha t)e^{-st}dt \\
&= \frac{1}{\alpha}\int_{\infty}^{\infty} x(\tau)e^{-s\tau/\alpha}d\tau && \text{(change variable } \tau = \alpha t) \\
&= \frac{1}{\alpha}\mathcal{L}(x, s/\alpha) \qquad \mathrm{Re}(s/\alpha) \in R.
\end{aligned}$$

Combining this with the time reversal property we obtain

$$\mathcal{L}\big(x(\alpha t), s\big) = \frac{1}{|\alpha|}\mathcal{L}(x, s/\alpha), \qquad a \neq 0, \mathrm{Re}(s/\alpha) \in R.$$

as required.

4.13. Consider the active electrical circuit from Figure 2.8 described by the differential equation from (2.2.3). Derive the transfer function of this system. Find an explicit system H that maps the input voltage x to the output voltage y. State whether this system is stable and/or regular. **Solution:** The differential equation modelling the circuit is

$$-\frac{x}{R_1} - C_1 Dx = \frac{y}{R_2} + C_2 Dy,$$

and taking Laplace transforms on both sides of this equation

$$\mathcal{L}y = -\frac{\frac{1}{R_1} + C_1 s}{\frac{1}{R_2} + C_2 s}\mathcal{L}(x) = -\frac{\alpha + \gamma s}{\beta + s}$$

where $\alpha = \frac{1}{R_1 C_2}$, $\beta = \frac{1}{R_2 C_2}$, and $\gamma = \frac{C_1}{C_2}$. The transfer function of the system mapping x to y is correspondingly

$$\lambda(H) = -\frac{\alpha + \gamma s}{\beta + s} = -\frac{\alpha}{\beta + s} - \frac{\gamma s}{\beta + s}$$

Applying partial fraction (as in Exercise 4.8) to the second term gives

$$\lambda(H) = -\frac{\alpha + \gamma\beta}{\beta + s} - \gamma$$

The first term $-\frac{\alpha+\gamma\beta}{\beta+s}$ corresponds with a regular system, say H_2, having impulse response

$$h_2 = -(\alpha + \gamma\beta)u(t)e^{-\beta t}$$

by using the Laplace transform pair from (4.2.3) with the integer $n = 0$. The term $-\gamma$ correspond with the system $H_1 = \gamma T_0$, i.e, the identity system multiplied by $-\gamma$. The system H that describes the mapping between input voltage x and output voltage y is thus

$$H(x) = H_1(x) + H_2(x) = -\gamma x + h_2 * x.$$

The system is not regular because the H_1 is not regular. The system is stable because H_1 is stable and H_2 is stable because the impulse response h_2 is absolutely integrable since $\beta = \frac{1}{R_2 C_2} > 0$. Equivalently the system is not regular because the transfer function does not have more poles than zero, and the system is stable because the transfer function has at least as many poles as zeros (equal in this case), and because all the poles lie strictly in the left half plane.

*4.14. Given the mass spring damper system described by (4.5.1), find the position signal p given that the force signal

$$f(t) = \Pi\big(t - \tfrac{1}{2}\big) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the rectangular function time shifted by $\frac{1}{2}$. Consider three cases:

(a) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \frac{\pi}{3}$,

(b) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \pi$,

(c) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = 2\pi$,

Plot the solution in each case, and comment on whether the system is underdamped, overdamped, or critically damped. **Solution:** Observe that the input force signal can be written as the sum of the step function u and its negated time-shift, that is,

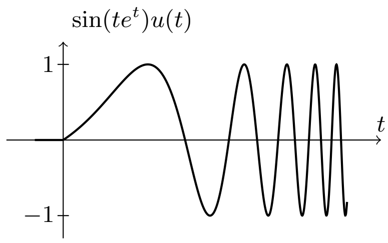
$$f(t) = u(t) - u(t-1) = u(t) - T_1 u(t)$$

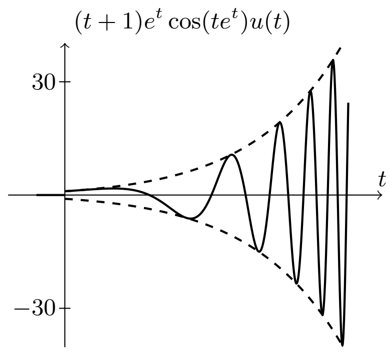
and so, the response of the linear, time invariant system H modelling the mass spring damper to input force signal f is

$$Hf = H(u - T_1 u) = Hu - T_1 Hu,$$

and so, $Hf(t) = Hu(t) - Hu(t-1)$, where Hu is the step response of the system. The step responses are described in Section 4.5. As described in Section 4.5, the system is underdamped when $B = \frac{\pi}{3}$, critically damped when $B = \pi$ and overdamped when $B = 2\pi$.

4.15. Plot the signal $x(t) = \sin(te^t)u(t)$ and find and plot its derivative Dx. Show that the region of convergence of x contains those complex numbers s with $\operatorname{Re} s > 0$ and that the region of convergence of Dx contains those with $\operatorname{Re} s > 1$. **Solution:** By application of the chain rule the derivative of $\sin(te^t)$ is $(t+1)e^t\cos(e^t)u(t)$. These signals are plotted below.





We have $|x(t)| = \left|\sin(te^t)u(t)\right| < 1$ for all $t \in \mathbb{R}$ and so

$$\begin{aligned}\mathcal{L}(x,s) &= \int_{-\infty}^{\infty} \sin(te^t)u(t)e^{-st}dt \\ &= \int_{0}^{\infty} \sin(te^t)e^{-st}dt \\ &\leq= \int_{0}^{\infty} \left|\sin(te^t)e^{-st}\right| dt \\ &\leq \int_{0}^{\infty} e^{-\operatorname{Re}(s)t}dt\end{aligned}$$

which is finite for all s with $\operatorname{Re}(s) > 0$ as required. For the derivative we have

$$\begin{aligned}\mathcal{L}Dx(s) &= \int_{-\infty}^{\infty} (t+1)e^t\cos(te^t)u(t)e^{-st}dt \\ &= \int_{0}^{\infty} (t+1)\cos(te^t)e^{-(s-1)t}dt \\ &\leq \int_{0}^{\infty} \left|(t+1)\cos(te^t)e^{-(s-1)t}\right| dt \\ &\leq \int_{0}^{\infty} (t+1)e^{-(\operatorname{Re}(s)-1)t}dt\end{aligned}$$

which is finite for all s with $\operatorname{Re}(s) > 1$ as required.

4.16. Show that the limit as $|s| \to 0$ of

$$\frac{e^{s/2} - e^{-s/2}}{s}$$

is equal to 1.

$*$4.17. Consider the mechanical system in Figure 2.5 from Exercise 2.2. After solving Exercise 2.2, find the transfer function of a linear shift-invariant H system mapping f to p. Now suppose that $M_1 = K_1 = K_2 = B = 1$ and $M_2 = 2$. Find the poles and zeros of H and draw a pole zero plot. Determine whether H is stable and/or regular. Find and plot the impulse response and the step response of H if they exist.

Solution: Let H be a linear time invariant system mapping f to p. The transfer function of H is

$$\lambda H(s) = \frac{1}{1 + 2s + 4s^2 + 2s^3 + 2s^4}.$$

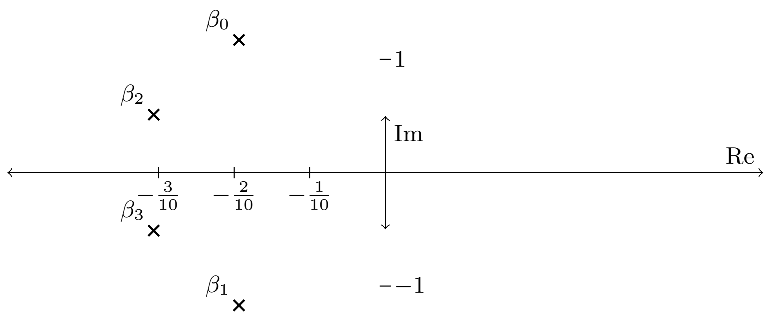
Factorising the polynomial on the denominator we obtain

$$\lambda H(s) = \frac{1}{(s - \beta_0)(s - \beta_1)(s - \beta_2)(s - \beta_3)}$$

where the roots are

$$\beta_0 = \beta_1^* = -0.193622 + 1.17046j, \quad \beta_2 = \beta_3^* = -0.306378 + 0.511255j$$

The system has no zeros and four poles. A pole zero plot is shown below.



Because there are atleast as many poles as zeros and the real part of all the poles is negative the system is stable. Because there are more poles than zeros the system is regular and has an impulse response. Applying partial fractions gives

$$\lambda H(s) = \frac{A_0}{s - \beta_0} + \frac{A_1}{s - \beta_1} + \frac{A_2}{s - \beta_2} + \frac{A_3}{s - \beta_3}$$

where

$$A_0 = A_1^* = -0.0887401 + 0.368434j, \quad A_2 = A_3^* = 0.0887401 + 0.863059j.$$

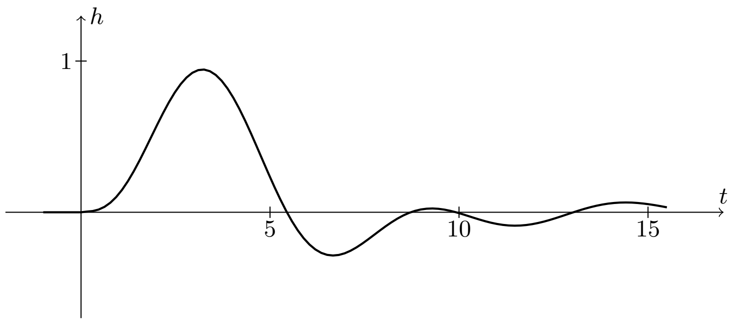
The impulse response is

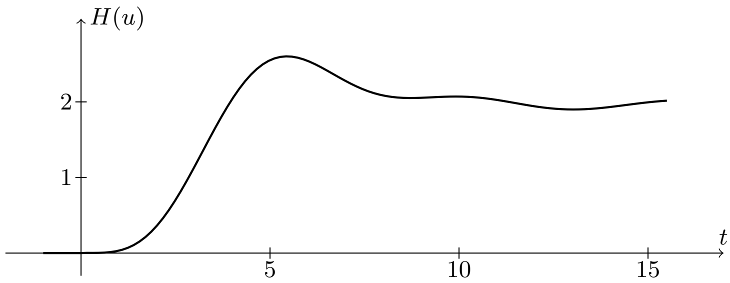
$$h(t) = u(t)\left(A_0 e^{\beta_0 t} + A_1 e^{\beta_1 t} + A_2 e^{\beta_2 t} + A_3 e^{\beta_3 t}\right).$$

The step response is

$$Hu = I_\infty h = u(t)\left(C_0 e^{\beta_0 t} + C_1 e^{\beta_1 t} + C_2 e^{\beta_2 t} + C_3 e^{\beta_3 t} - B\right)$$

where $C_n = A_n/\beta_n$, $n = 0, 1, 2, 3$ and $B = C_0 + C_1 + C_2 + C_3$. These responses are plotted below.





4.18. Consider the electromechanical system in Figure 2.6 from Exercise 2.3. After solving Exercise 2.3, find the transfer function of a linear shift-invariant system that maps the input voltage v to the motor angle θ. Under the assumption that the motor coefficients satisfy $L = 0$ and $K_b = K_\tau = B = R = J = 1$ draw a pole zero plot and determine whether this system is stable and/or regular. Find and plot the impulse response and step response if they exist.

Solution: Exercise 2.3 finds the following differential equation relating v and θ,

$$v = \theta + \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta.$$

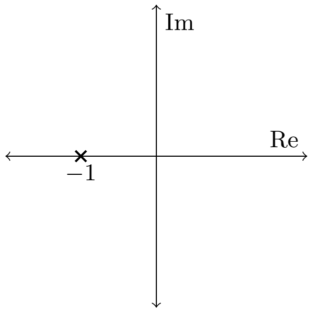
The transfer function is

$$\frac{1}{1 + \left(\frac{RB}{K_\tau} + K_b\right) s + \frac{RJ}{K_\tau} s^2}.$$

Under the assumption that $K_b = K_\tau = B = R = J = 1$ the transfer function is

$$\frac{1}{1 + 2s + s^2} = \frac{1}{(1+s)^2}.$$

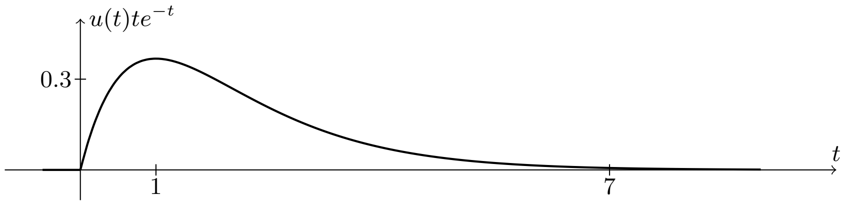
There are two equal real poles at $s = -1$ and no zeros. A pole zero plot is below.

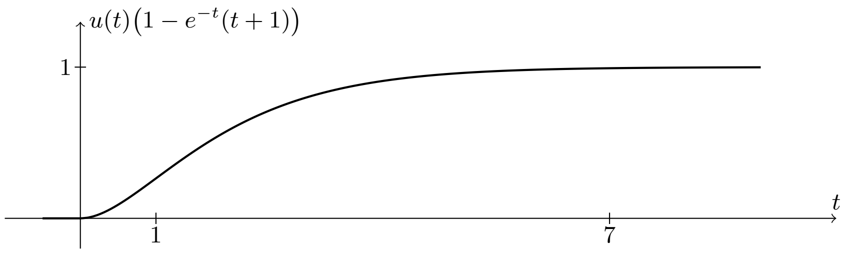


The system is regular because there are more poles than zeros. The system is stable because there are at least as many poles as zeros and all poles have negative real part. The impulse response is found to be $h(t)u(t)te^{-t}$ by application of the inverse Laplace transform. The step response is given by application of the integrator system

$$Hu = I_\infty(h) = \int_{-\infty}^{t} u(\tau)te^{-\tau}d\tau = \int_{0}^{t} te^{-\tau}d\tau = u(t)\big(1 - e^{-t}(t+1)\big).$$

These responses are plotted below





$**$4.19. Let x be a signal. Show that the complex exponential signal $e^{st} \in \operatorname{dom} x$ if and only if the signal $x(t)e^{-st}$ is absolutely integrable. **Solution:** The set $\operatorname{dom} x$ contains all those signals such that

$$\int_{-\infty}^{\infty} |x(\tau)x(t-\tau|\, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

If $e^{st} \in \operatorname{dom} x$ then

$$\int_{-\infty}^{\infty} \left|x(\tau)e^{s(t-\tau)}\right| d\tau = \left|e^{st}\right| \int_{-\infty}^{\infty} \left|x(\tau)e^{-s\tau}\right| d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Setting $t = 0$ we find that

$$\int_{-\infty}^{\infty} \left|x(\tau)e^{-s\tau}\right| d\tau < \infty$$

and so $x(t)e^{-st}$ is absolutely integrable. On the other hand, if $x(t)e^{-st}$ is absolutely integrable then

$$\int_{-\infty}^{\infty} \left| x(\tau)e^{s(t-\tau)} \right| d\tau = \left| e^{st} \right| \|x(t)e^{-st}\|_1 < \infty \qquad \text{for all } t \in \mathbb{R}.$$

since e^{st} is finite for all t, and so, $e^{st} \in \operatorname{dom} x$.

$**$4.20. Show that the complex exponential signal $e^{st} \in \operatorname{dom} f\ g$ if and only if $s \in \operatorname{roc} f \cap \operatorname{roc} g$, that is, $\operatorname{cep}\operatorname{dom} f\ g = \operatorname{roc} f \cap \operatorname{roc} g$.

Chapter 5

The Fourier transform

Exercises

5.1. Plot the signal $e^{-\alpha|t|}$ where $\alpha > 0$ and find its Fourier transform.
Solution:

$$\begin{aligned}
\mathcal{F}(e^{-\alpha|t|}) &= \int_{-\infty}^{\infty} e^{-\alpha|t|}e^{-j2\pi ft}dt \\
&= \int_{0}^{\infty} e^{-\alpha t}e^{-j2\pi ft}dt + \int_{-\infty}^{0} e^{\alpha t}e^{-j2\pi ft}dt \\
&= \int_{0}^{\infty} e^{-(j2\pi f+\alpha)t}dt + \int_{-\infty}^{0} e^{-(j2\pi f-\alpha)t}dt \\
&= \left[\frac{e^{-(j2\pi f+\alpha)t}}{-(j2\pi f+\alpha)}\right]_0^{\infty} + \left[\frac{e^{-(j2\pi f-\alpha)t}}{-(j2\pi f-\alpha)}\right]_{-\infty}^{0}.
\end{aligned}$$

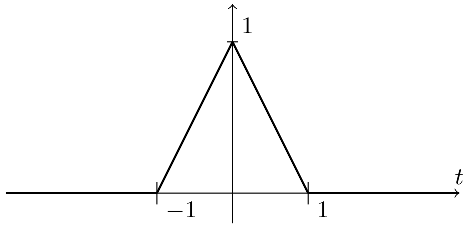
Because $\alpha > 0$, the limits as $t \to \infty$ and $t \to -\infty$ go to zero leaving

$$\frac{1}{j2\pi f+\alpha} - \frac{1}{j2\pi f-\alpha} = \frac{j2\pi f+\alpha-j2\pi f+\alpha}{(j2\pi f+\alpha)(j2\pi f-\alpha)} = \frac{2\alpha}{4\pi^2 f^2+\alpha^2}.$$

5.2. Plot the signal

$$\triangle(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform. **Solution:** This signal is often called the **triangle function** or **triangle pulse**.



You can do this directly using the formula for the Fourier transform and integrating by parts. However, it is easier to first realise that the triangle pulse is the convolution of the rectangular function with itself. That is $\Pi * \Pi = \triangle$. To see this write

$$(\Pi * \Pi)(t) = \int_{-\infty}^{\infty} \Pi(\tau)\Pi(t-\tau)d\tau = \int_{-1/2}^{1/2} \Pi(t-\tau)d\tau$$

Now $\Pi(t-\tau) = 1$ for τ in the interval $(-\frac{1}{2}+t, \frac{1}{2}+t)$ and zero otherwise. Thus, the integral evaluates to zero if $t \geq 1$ or $t \leq -1$. When $t \in (-1, 0]$

$$(\Pi * \Pi)(t) = \int_{-1/2}^{1/2+t} d\tau = t + 1$$

and when $t \in [0, 1)$

$$(\Pi * \Pi)(t) = \int_{-1/2+t}^{1/2} d\tau = 1 - t$$

as required. Now, by the convolution theorem (5.0.3)

$$\mathcal{F}(\Pi * \Pi) = \mathcal{F}\triangle = \mathcal{F}\Pi\,\mathcal{F}\Pi = \operatorname{sinc}^2(t).$$

∗5.3. Show that the sinc function is square integrable, but not absolutely integrable. **Solution:** Our proof is by contradiction. Assume that sinc is absolutely integrable. Then

$$\begin{aligned}
\|\operatorname{sinc}\|_1 &= \int_{-\infty}^{\infty} |\operatorname{sinc}(t)|\, dt \\
&> \int_{0}^{\infty} |\operatorname{sinc}(t)|\, dt \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} \left|\frac{\sin(\pi t)}{\pi t}\right| dt \\
&= \sum_{n=1}^{\infty} a_n
\end{aligned}$$

where we put

$$a_n = \int_{n-1}^{n} \left|\frac{\sin(\pi t)}{\pi t}\right| dt.$$

Under our assumption that sinc is absolutely integrable we must have that the infinite sum $a_1 + a_2 + \dots$ converge to a finite number. Now

$$a_n \geq \int_{n-1}^{n} \left|\frac{\sin(\pi t)}{\pi n}\right| dt = \frac{1}{\pi n}\int_{n-1}^{n} |\sin(\pi t)|\, dt = \frac{2}{\pi^2 n}.$$

However, the sum

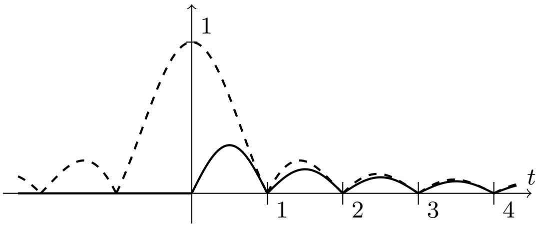
$$\sum_{n=1}^{\infty} a_n = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}$$

involves the harmonic series (a p-series with $p = 1$) and so diverges (to show this use either an integral test or the condensation test). Thus, our initial hypothesis that sinc is absolutely integrable is false.

Graphically, the argument we have used bounds $|\mathrm{sinc}|$ above the function

$$b(t) = \begin{cases} 0 & t \leq 0 \\ \left|\frac{\sin(\pi t)}{\pi n}\right| & t \in (n-1, n] \end{cases}$$

and then shows that b is not absolutely integrable. The function $|\mathrm{sinc}|$ (dashed) and b (solid) are plotted in the figure below.



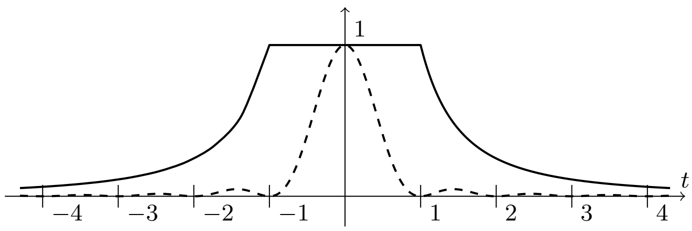
To show that sinc is square integrable observe that $\mathrm{sinc}^2(t)$ is bounded below the function

$$g(t) = \begin{cases} 1 & |t| \leq 1 \\ \frac{1}{t^2} & \text{otherwise,} \end{cases}$$

that is $\mathrm{sinc}^2(t) \leq g(t)$ for all $t \in \mathbb{R}$. Thus

$$\begin{aligned} \| \mathrm{sinc} \|_2 &= \int_{-\infty}^{\infty} |\mathrm{sinc}(t)|^2 \, dt \\ &\leq \int_{-\infty}^{\infty} g(t) dt \\ &= \int_{-1}^{1} dt + 2 \int_{1}^{\infty} \frac{1}{t^2} \\ &= 2 - \left[\frac{1}{t} \right]_1^{\infty} = 2 + 1 = 3. \end{aligned}$$

The figure below plots sinc^2 (dashed) and the bounding function g.



∗5.4. Show the the magnitude spectrum of the normalised Butterworth filter B_m satisfies

$$|\Lambda B_m(f)| = \sqrt{\frac{1}{f^{2m} + 1}}.$$

Solution: Recall that the transfer function of B_m is

$$\lambda B_m(s) = \frac{1}{\prod_{i=1}^m (\frac{s}{2\pi} - \beta_i)} = \frac{(2\pi)^m}{\prod_{i=1}^m (s - 2\pi\beta_i)},$$

where $\beta_1, \dots, \beta_m$ are the roots of the polynomial $s^{2m} + (-1)^m$ that lie strictly in the left half of the complex plane (have negative real part). Specifically, these roots are

$$\beta_k = \begin{cases} \exp\left(j\frac{\pi}{2}(1 + \frac{2k-1}{m})\right), & k = 1, \dots, m \\ \exp\left(j\frac{\pi}{2}(1 - \frac{2k-1}{m})\right), & k = m+1, \dots, 2m \end{cases}$$

or equivalently

$$\beta_k = \begin{cases} j\cos\left(\frac{\pi(2k-1)}{2m}\right) - \sin\left(\frac{\pi(2k-1)}{2m}\right), & k = 1,\dots,m \\ j\cos\left(\frac{\pi(2k-1)}{2m}\right) + \sin\left(\frac{\pi(2k-1)}{2m}\right), & k = m+1,\dots,2m. \end{cases}$$

Observe that the roots $\beta_{m+1},\dots,\beta_{2m}$ are given by negating the real parts of $\beta_1,\dots,\beta_m$, that is, $\beta_{m+i} = j(\beta_i/j)^*$. The squared magnitude of the polynomial on the denominator is

$$\begin{aligned} \left|\prod_{i=1}^{m}(jf-\beta_i)\right|^2 &= \left(\prod_{i=1}^{m}(jf-\beta_i)\right)\left(\prod_{i=1}^{m}(jf-\beta_i)\right)^* \\ &= \prod_{i=1}^{m}(jf-\beta_i)(jf-\beta_i)^* \\ &= \prod_{i=1}^{m}(jf-\beta_i)j^*(f-(\beta_i/j)^*) \end{aligned}$$

and because $j^*/j = -1$ we have

$$\begin{aligned} \left|\prod_{i=1}^{m}(jf-\beta_i)\right|^2 &= (-1)^m\prod_{i=1}^{m}(jf-\beta_i)(jf-j(\beta_i/j)^*) \\ &= (-1)^m\prod_{i=1}^{m}(jf-\beta_i)(jf-\beta_{m+i}) \\ &= (-1)^m\prod_{i=1}^{2m}(jf-\beta_i). \end{aligned}$$

Because $\beta_1,\dots,\beta_{2m}$ are the roots of the polynomial $s^{2m}+(-1)^m$ we have

$$\left|\prod_{i=1}^{m}(jf-\beta_i)\right|^2 = (-1)^m\left((jf)^{2m}+(-1)^m\right) = f^{2m}+1.$$

It follows that the magnitude spectrum of B_m is

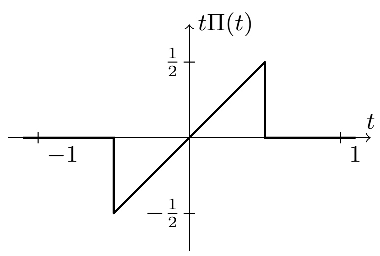
$$|\Lambda(B_m)| = \sqrt{\frac{1}{f^{2m}+1}}.$$

∗5.5. Find and plot the impulse response of the normalised lowpass Butterworth filters B_1, B_2 and B_3.

5.6. Plot the signal

$$t\Pi(t) = \begin{cases} t & -\frac{1}{2} < t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform. **Solution:**



A direct approach is

$$\mathcal{L}(t\Pi(t)) = \int_{-\infty}^{\infty} t\Pi(t)e^{-sft}dt = \int_{-1/2}^{1/2} te^{-st}dt$$

and integrating by parts gives

$$\begin{aligned}
\mathcal{L}(t\Pi(t)) &= \left[t\frac{e^{-st}}{-s}\right]_{-1/2}^{1/2} - \int_{-1/2}^{1/2} \frac{e^{-st}}{-s}dt \\
&= \frac{e^{-s/2}+e^{s/2}}{-2s} - \left[\frac{e^{-st}}{s^2}\right]_{-1/2}^{1/2} \\
&= \frac{e^{-s/2}+e^{s/2}}{-2s} - \frac{e^{-s/2}-e^{s/2}}{s^2} \\
&= \frac{1}{s}\left(-\frac{e^{s/2}+e^{-s/2}}{2} + \frac{e^{s/2}-e^{-s/2}}{s}\right).
\end{aligned}$$

Because $t\Pi(t)$ is absolutely integrable its region of convergence includes the imaginary axis and we can obtain the Fourier transform by evaluating the Laplace transform at $s = j2\pi f$,

$$\begin{aligned}
\mathcal{F}(t\Pi(t))(f) &= \mathcal{L}x(j2\pi f) \\
&= \frac{1}{2\pi jf}\left(-\frac{e^{j\pi f}+e^{-j\pi f}}{2} + \frac{e^{j\pi f}-e^{-j\pi f}}{2j\pi f}\right) \\
&= \frac{1}{2\pi jf}\left(\operatorname{sinc}(f) - \cos(\pi f)\right).
\end{aligned}$$

An alternative approach is to observe that

$$\mathcal{F}D\operatorname{sinc}(f) = \Lambda D\mathcal{F}\operatorname{sinc}(f) = j2\pi f\Pi(f),$$

and so, by duality,

$$\mathcal{F}(j2\pi t\Pi(t))(f) = \mathcal{F}\mathcal{F}D\operatorname{sinc}(f) = D\operatorname{sinc}(-f)$$

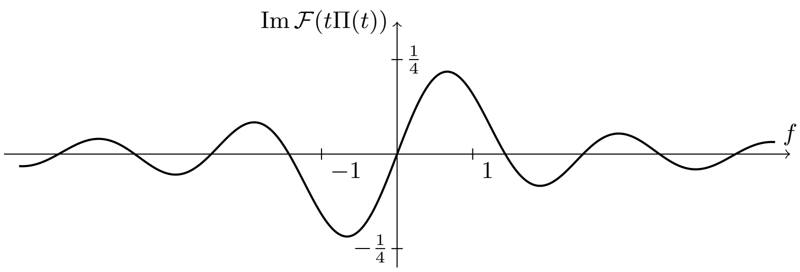
The derivative of the sinc function is given in (2.2.5)

$$D\operatorname{sinc}(-f) = \frac{1}{\pi f^2}\big(\sin(\pi f) - \pi f\cos(\pi f)\big) = \frac{1}{f}\big(\operatorname{sinc}(f) - \cos(\pi f)\big).$$

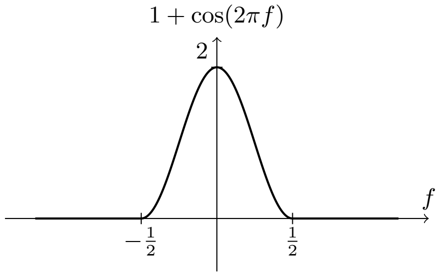
Dividing by $j2\pi$ we obtain

$$\mathcal{F}(t\Pi(t))(f) = \frac{1}{2j\pi^2 f^2}\big(\pi f\cos(\pi f) - \sin(\pi f)\big) = \frac{1}{2j\pi f}\big(\operatorname{sinc}(f) - \cos(\pi f)\big)$$

again. A plot of the Fourier transform is below. Observe that the Fourier transform is purely imaginary so we plot the imaginary part.



5.7. Let x be the signal with Fourier transform $\hat{x}(f) = \Pi(f)\big(\cos(2\pi f)+1\big)$. Plot the Fourier transform $\hat{x}$ and find and plot x. **Solution:**



We will solve the problem in two ways. Firstly, by directly application of the inverse Fourier transform and secondly, by applications of the duality and modulation properties. Application of the inverse Fourier transform gives

$$x = \mathcal{F}^{-1}\big(\Pi(f) + \Pi(f)\cos(2\pi f)\big) = \mathcal{F}^{-1}\big(\Pi(f)\big) + \mathcal{F}^{-1}\big(\Pi(f)\cos(2\pi f)\big).$$

Now

$$\begin{aligned}\mathcal{F}^{-1}\big(\Pi(f)\big) &= \int_{-\infty}^{\infty} \Pi(f)e^{2\pi jft}df \\ &= \int_{-1/2}^{1/2} e^{2\pi jft}df \\ &= \frac{e^{\pi jt} - e^{-\pi jt}}{2\pi jt} = \frac{2j\sin(\pi t)}{2\pi jt} = \frac{\sin(\pi t)}{\pi t} = \operatorname{sinc}(t)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^{-1}\big(\Pi(f)\cos(2\pi f)\big) &= \int_{-\infty}^{\infty} \Pi(f)\cos(2\pi f)e^{2\pi jft}df \\ &= \int_{-1/2}^{1/2} \cos(2\pi f)e^{2\pi jft}df \\ &= \int_{-1/2}^{1/2} \tfrac{1}{2}(e^{2\pi jf} + e^{-2\pi jf})e^{2\pi jft}df \\ &= \int_{-1/2}^{1/2} e^{2\pi jf(t+1)}df + \tfrac{1}{2}\int_{-1/2}^{1/2} e^{2\pi jf(t-1)}df \\ &= \tfrac{1}{2}\operatorname{sinc}(t+1) + \tfrac{1}{2}\operatorname{sinc}(t-1)\end{aligned}$$

by working similarly to the previous equation. Putting these together we obtain the time domain signal

$$x(t) = \operatorname{sinc}(t) + \tfrac{1}{2}\operatorname{sinc}(t+1) + \tfrac{1}{2}\operatorname{sinc}(t-1).$$

We now derive the same result using duality (5.1.5) and the modulation (5.0.7) properties. By duality

$$x(-f) = \mathcal{F}\big(\Pi(f)\big) + \mathcal{F}\big(\Pi(f)\cos(2\pi f)\big)$$

Because $\mathcal{F}(\Pi) = \operatorname{sinc}$ the time shift properties yields

$$\mathcal{F}\big(\Pi(t)\big) = \operatorname{sinc}(t).$$

Now, by the modulation property

$$\begin{aligned}\mathcal{F}\big(\Pi(t)\cos(2\pi t), f\big) &= \tfrac{1}{2}\mathcal{F}\big(\Pi(t), f-1\big) + \tfrac{1}{2}\mathcal{F}\big(\Pi(t), f+1\big) \\ &= \tfrac{1}{2}\operatorname{sinc}(f-1) + \tfrac{1}{2}\operatorname{sinc}(f+1).\end{aligned}$$

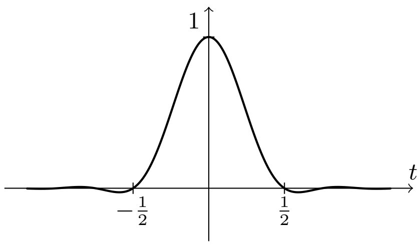
Now

$$x(-f) = \operatorname{sinc}(f) + \tfrac{1}{2}\operatorname{sinc}(f-1) + \tfrac{1}{2}\operatorname{sinc}(f+1)$$

Putting $t = -f$

$$x(t) = \operatorname{sinc}(t) + \tfrac{1}{2}\operatorname{sinc}(t+1) + \tfrac{1}{2}\operatorname{sinc}(t-1)$$

because sinc is even. A plot of the Fourier transform is below. The shape of the Fourier transform is somewhat sinc-like, but the oscillations decay faster as $|t| \to \infty$.



5.8. State whether the following signals are bandlimited and, if so, find the bandwidth:

(a) $\operatorname{sinc}(4t)$,
(b) $\Pi(t/4)$,
(c) $\cos(2\pi t)\operatorname{sinc}(t)$,
(d) $e^{-|t|}$.

Solution: Let $S_\alpha(x,t) = x(\alpha t)$ be the time scaler system. We have

$$\begin{aligned}\mathcal{F}\big(S_\alpha(x), f\big) &= \int_{-\infty}^{\infty} x(\alpha t)e^{-2\pi j t}dt \\ &= \frac{1}{\alpha}\int_{-\infty}^{\infty} x(\gamma)e^{-2\pi j\gamma/\alpha}d\gamma \qquad \text{(ch. var. } \gamma = \alpha t) \\ &= \frac{1}{\alpha}\mathcal{F}\big(x, f/\alpha\big) \\ &= \frac{1}{\alpha}S_{1/\alpha}\big(\mathcal{F}(x), f\big).\end{aligned}$$

The Fourier transform of $S_\alpha(\operatorname{sinc})(t) = \operatorname{sinc}(4t)$ is

$$\mathcal{F}\big(\operatorname{sinc}(4t)\big) = \tfrac{1}{4}\Pi(f/4),$$

and the signal is bandlimited with bandwidth 2 because $\Pi(f/4) = 0$ whenever $|f| > 2$. By duality

$$\tfrac{1}{4}\mathcal{F}(\Pi(f/4)) = \operatorname{sinc}(4t)$$

and so $\mathcal{F}(\Pi(f/4)) = 4\operatorname{sinc}(4t)$. This signal is not bandlimited because the sinc function is unbounded in time. By the modulation property of Fourier transform (5.0.7),

$$\mathcal{F}\big(\cos(2\pi t)\operatorname{sinc}(t), f\big) = \mathcal{F}(\operatorname{sinc}, f-1) + \mathcal{F}(\operatorname{sinc}, f+1) = \Pi(f-1) + \Pi(f+1).$$

This is bandlimited with bandwidth $\frac{3}{2}$. In Exercise 5.1 we showed that

$$\mathcal{F}(e^{-|t|}) = \frac{2}{4\pi^2 f^2 + 1}.$$

This signal is not bandlimited.

5.9. Show that

$$\sum_{n\in\mathbb{Z}} e^{\alpha|n|} = 1 + \frac{2}{e^{-\alpha} - 1}$$

if $\alpha < 0$ (Hint: solve Exercise 3.7 first). **Solution:** Put $r = e^{\alpha}$. Because $\alpha < 0$ the series $r + r^2 + r^3 + \dots$ converges absolutely and so

$$\sum_{n\in\mathbb{Z}} e^{\alpha|n|} = \sum_{n\in\mathbb{Z}} r^{|n|} = 1 + 2\sum_{n=1}^{\infty} r^n$$

The sum in the equation on the right is a geometric series evaluating to

$$\frac{r}{1-r} = \frac{e^{\alpha}}{1-e^{\alpha}} = \frac{1}{e^{-\alpha} - 1}.$$

5.10. Show that if a sequence is absolutely summable then it is also square summable. **Solution:** Suppose that the discrete time signal a is absolutely summable so that $\|a\|_1 = \sum_{n\in\mathbb{Z}} |a_n| < \infty$. Let A be the subset of $\mathbb{Z}$ such that $|a_n| \geq 1$ whenever $n \in A$. That is

$$A = \{n \; ; \; |a_n| \geq 1\}.$$

Let $|A|$ denote the number of elements in the set A. We have

$$\infty > \|a\|_1 = \sum_{n\in\mathbb{Z}} |a_n| \geq \sum_{n\in A} |a_n| \geq |A|$$

and so, A contains a finite number of elements. Thus

$$B = \sum_{n\in A} |a_n|^2 < \infty$$

Now, $|a_n|^2 < |a_n| < 1$ for all $n \notin A$ and so

$$C = \sum_{n\notin A} |a_n|^2 \leq \sum_{n\notin A} |a_n| \leq \|a\|_1 < \infty.$$

Thus

$$\|a\|_2^2 = \sum_{n\in\mathbb{Z}} |a_n|^2 = \sum_{n\in A} |a_n|^2 + \sum_{n\notin A} |a_n|^2 = B + C < \infty.$$

5.11. Show that $\sum_{k=0}^{N-1} e^{j2\pi nk/N}$ is equal to N if n is a multiple of N and zero if n is any integer not a multiple of N. (Hint: use the result from Exersise 3.7) **Solution:** First observe when is a multiple of N,

$$\sum_{k=0}^{N-1} e^{j2\pi nk/N} = \sum_{k=0}^{N-1} 1 = N.$$

It remains to show that the sum is zero if n is an integer not a multiple of N. It is helpful to reparameterise in order make the connection with Exercise 3.7. We have

$$\sum_{k=0}^{N-1} e^{j2\pi nk/N} = \sum_{\ell=1}^{N} e^{\beta\ell}$$

where $\ell = k+1$ and $\beta = j2\pi n/N$. From Exercise 3.7, this sum is equal to

$$\frac{e^{\beta(N+1)} - e^{\beta}}{e^{\beta} - 1}.$$

The numerator is equal to

$$e^{\beta(N+1)} - e^{\beta} = e^{\beta} e^{\beta N/2}(e^{\beta N/2} - e^{-\beta N/2}) = e^{\beta} e^{\beta N/2} 2j \sin(\alpha N)$$

where $\alpha = \pi n/N$ so that $\beta = j2\alpha$. The denominator is equal to

$$e^{\beta} - 1 = e^{\beta/2}(e^{\beta/2} - e^{-\beta/2}) = e^{\beta/2} 2j \sin(\alpha).$$

Using these expression for the numerator and denominator we obtain

$$\sum_{k=0}^{N-1} e^{j2\pi nk/N} = e^{(N+1)\beta/2} \frac{\sin(\alpha N)}{\sin(\alpha)} = e^{j\pi n(N+1)/N} \frac{\sin(\pi n)}{\sin(\pi n/N)}.$$

If n is an integer not a multiple of N then $\sin(\pi n/N) \neq 0$ while $\sin(\pi n) = 0$ and so

$$\sum_{k=0}^{N-1} e^{j2\pi nk/N} = e^{j\pi n(N+1)/N} \frac{\sin(\pi n)}{\sin(\pi n/N)} = 0$$

as required.

5.12. Let $d = \mathcal{D}_N c$ be the discrete Fourier transform of the sequence c. Show that

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} \qquad n = 0, \dots, N-1.$$

(Hint: use the result from Exersize 5.11) **Solution:** We have

$$d_k = \mathcal{D}_N(c, k) = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/N}$$

and so

$$\begin{aligned}
\frac{1}{N} \sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} c_m e^{-j2\pi mk/N} \right) e^{j2\pi nk/N} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} c_m e^{j2\pi(n-m)k/N} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} c_m \sum_{k=0}^{N-1} e^{j2\pi(n-m)k/N}.
\end{aligned}$$

The integers n and m are from the set $\{0, \dots, N-1\}$ and so the difference $n-m$ takes values from the set $\{-N+1, \dots, N-1\}$. From Exersise 5.11 the inner sum satisfies

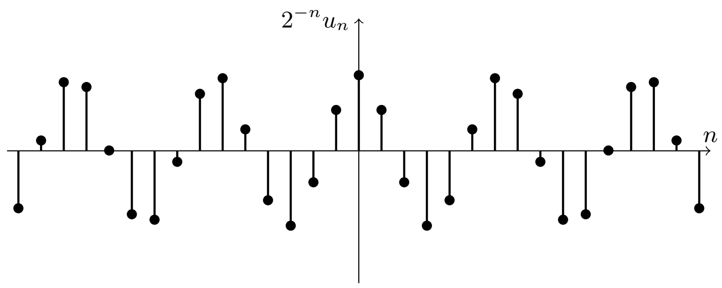
$$\delta_{n-m} = \sum_{k=0}^{N-1} e^{j2\pi(n-m)k/N} = \begin{cases} N & n-m=0 \\ 0 & n-m \neq 0 \end{cases}$$

and so

$$\frac{1}{N}\sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} = \frac{1}{N}\sum_{m=0}^{N-1} c_m \delta_{n-m} = c_n \qquad n = 0, \dots, N-1.$$

as required.

5.13. Plot the sequence $\cos(n)$ and determine whether it is bounded or periodic. **Solution:** A plot is below.



The sequence is bounded below any number greater than 1 because $|\cos(x)| \leq 1$ (precisely 1 would also work in this case). The sequence is not periodic. To see this, suppose the sequence has period $T \in \mathbb{Z}$ so that $\cos(n) = \cos(n + kT)$ for all $k \in \mathbb{Z}$. The period of $\cos(x)$ is 2π and $\cos(x) = \cos(y)$ only if $x = y + 2\pi\ell$ for some $\ell \in \mathbb{Z}$. Thus, we must have $n = n + kT + 2\pi\ell$ for some ℓ and each k. But, now $\pi = \frac{kT}{2\ell}$ which violates the fact that π is irrational. Thus, no such period T exists.

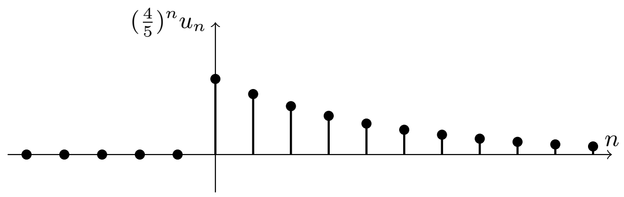
5.14. Find the discrete time Fourier transform of the sequence $\alpha^n u_n$ where $|\alpha| < 1$ and u_n is the step sequence. Plot the sequence and the magnitude of the discrete time Fourier transform when $\alpha = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}$. **Solution:** The discrete time Fourier transform is

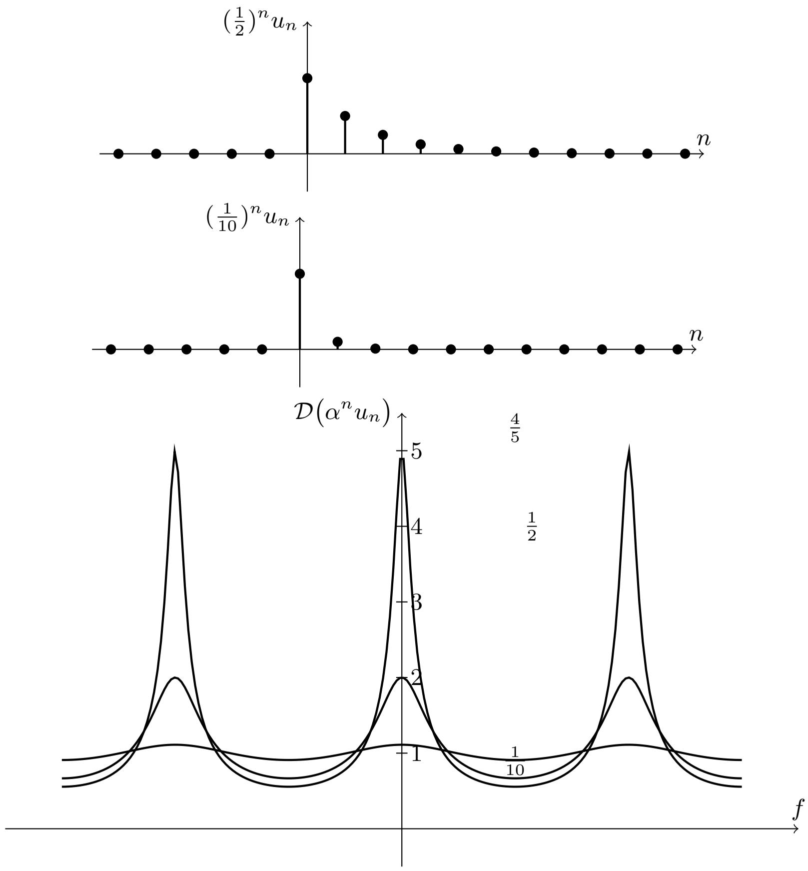
$$\begin{aligned}\mathcal{D}(\alpha^n u_n) &= \sum_{n\in\mathbb{Z}} \alpha^n u_n e^{-j2\pi nf} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j2\pi nf} \\ &= \sum_{n=0}^{\infty} (\alpha e^{-j2\pi f})^n = \frac{1}{1 - \alpha e^{-j2\pi f}}\end{aligned}$$

by the formula for the sum of a geometric progression. The sum converges because $|\alpha| < 1$. In the case that α is real the magnitude of the discrete time Fourier transform is

$$|\mathcal{D}(\alpha^n u_n)| = \sqrt{\frac{1}{1 - 2\alpha\cos(2\pi f) + \alpha^2}}.$$

Plots of the sequence and the discrete time Fourier transform for $\alpha = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}$ is below.





5.15. Given (5.1.3) show that (5.1.4) holds. **Solution:** From (5.1.3) we have that

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, dt = \|\hat{x}\|_2^2.$$

Denote by

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

typically referred to as the **inner product** of signals x and y. We have

$$\|x\|_2^2 = \langle x, x \rangle = \|\hat{x}\|_2^2 = \langle \hat{x}, \hat{x} \rangle.$$

Replacing x with $x + y$ we have

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 = \|\hat{x} + \hat{y}\|^2 = \|\hat{x}\|^2 + 2\langle \hat{x}, \hat{y} \rangle + \|\hat{y}\|^2.$$

And since $\|x\|_2^2 = \|\hat{x}\|_2^2$ and $\|y\|_2^2 = \|\hat{y}\|_2^2$ as a result of (5.1.3) we find that

$$\langle x, y \rangle = \langle \hat{x}, \hat{y} \rangle$$

which is precisely (5.1.4).

$**$5.16. Let x and y be square integrable signals. Show that $\mathcal{F}(xy) = \hat{x} * \hat{y}$. **Solution:** See `http://math.stackexchange.com/questions/605232/fourier-transform-of-convolution-for-l2-fur`

∗∗5.17. Let c be an absolutely summable sequence. Show that

$$\mathcal{F}\sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(t-n) = \sum_{n\in\mathbb{Z}} c_n \mathcal{F}(\operatorname{sinc}(t-n)).$$

∗∗5.18. Let c be a square summable sequence and let

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(t-n)$$

be the bandlimited signal with samples $x(n) = c_n$. Show that

$$\mathcal{F}x = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \mathcal{F}\big(\operatorname{sinc}(t-n)\big) \quad \text{a.e.}$$

Find an example where equality does not hold pointwise.

∗∗5.19. Let x be an absolutely integrable signal. Show that the periodic summation $\sum_{m\in\mathbb{Z}} x(t+mP)$ is a locally integrable signal. Show that this is not necessarily true if x is square integrable, but not absolutely integrable.

Chapter 6

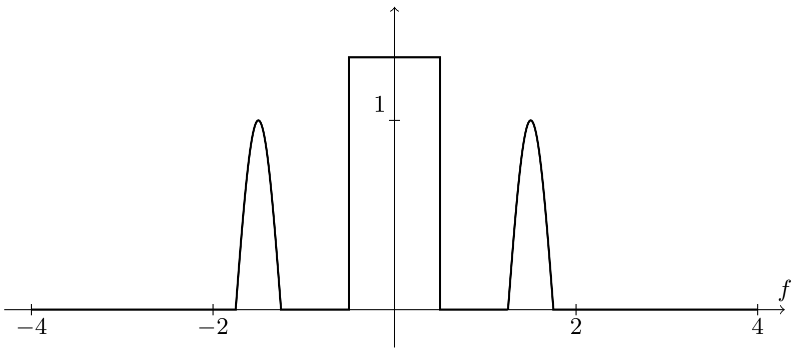
Discrete time systems

Exercises

6.1. Let x be the signal with Fourier transform

$$\hat{x}(t) = \tfrac{4}{3}\Pi(f) - \cos(2\pi f)\big(\Pi(2f-3) + \Pi(2f+3)\big).$$

Plot the Fourier transform. Find and plot x. **Solution:** A plot of the Fourier transform is below.



The time domain signal x can be found by direct application of the inverse Fourier transform, but a simpler approach uses the time shifting, time scaling properties, and modulation properties of the Fourier transform (Section 5). Let a be the signal with Fourier transform $\hat{a}(f) = \Pi(2f)$. The time scaling property asserts that

$$a(t) = \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{t}{2}\right).$$

From the modulation property of the Fourier transform

$$\mathcal{F}\big(\cos(3\pi t)a(t)\big) = \tfrac{1}{2}\hat{a}(f-\tfrac{3}{2}) + \tfrac{1}{2}\hat{a}(f+\tfrac{3}{2}) = \tfrac{1}{2}\Pi(2f-3) + \tfrac{1}{2}\Pi(2f+3)$$

Let $b(t) = \cos(3\pi t)a(t)$ so that $\hat{b}(f) = \frac{1}{2}\Pi(2f-3) + \frac{1}{2}\Pi(2f+3)$. Now, put

$$\begin{aligned} c(t) &= T_{-1}b(t) + T_1 b(t) \\ &= b(t+1) + b(t-1) \\ &= \cos(3\pi t + 3\pi)a(t+1) + \cos(3\pi t - 3\pi)a(t-1) \end{aligned}$$

and because

$$-\cos(3\pi t) = \cos(3\pi t + 3\pi) = \cos(3\pi t - 3\pi)$$

we have

$$c(t) = -\cos(3\pi t)\big(a(t+1) + a(t-1)\big).$$

From the time shift property of the Fourier transform

$$\begin{aligned}\hat{c}(f) &= \mathcal{F}(T_{-1}b + T_1 b)\\ &= e^{2\pi f j}\hat{b}(f) + e^{-2\pi f j}\hat{b}(f)\\ &= 2\cos(2\pi f)\hat{b}(f)\\ &= \cos(2\pi f)\big(\Pi(2f-3) + \Pi(2f+3)\big).\end{aligned}$$

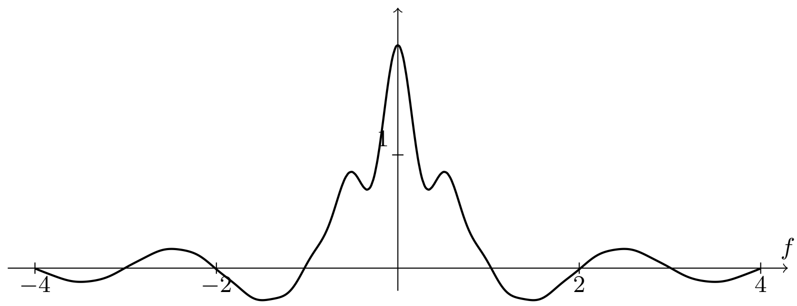
It remains to observe that

$$\hat{x}(f) = \tfrac{4}{3}\Pi(f) - \hat{c}(f)$$

and so

$$\begin{aligned}x(t) &= \tfrac{4}{3}\operatorname{sinc}(f) - c(t)\\ &= \tfrac{4}{3}\operatorname{sinc}(f) + \cos(3\pi t)\big(a(t+1) + a(t-1)\big)\\ &= \tfrac{4}{3}\operatorname{sinc}(f) + \tfrac{1}{2}\cos(3\pi t)\left(\operatorname{sinc}\left(\tfrac{t+1}{2}\right) + \operatorname{sinc}\left(\tfrac{t-1}{2}\right)\right)\end{aligned}$$

This signal is plotted below.



6.2. Find the Fourier transform of the Blackman window (6.4.1).

6.3. Show that the z-transform of the sequence $a^n u_n$ is $z/(z-a)$ with region of convergence $|z| > |a|$. **Solution:** The z-transform is

$$\mathcal{Z}(a^n u_n) = \sum_{n\in\mathbb{Z}} a^n u_n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^{-n}.$$

This sum is a geometric progression that converges to

$$\frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

when $|z/a| > 1$ and diverges otherwise. The region of convergence is thus $|z| > |a|$.

6.4. Show that the z-transform of the sequence $[n]_k u_n$ where $[n]_k = n(n-1)\dots(n-k+1)$ is a falling factorial is

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1.$$

Solution: Observe that

$$\begin{aligned}\mathcal{Z}\big([n]_k u_n\big) &= \sum_{n\in\mathbb{Z}} [n]_k u_n z^{-n}\\ &= \sum_{n=0}^{\infty} [n]_k z^{-n}\\ &= \sum_{n=-1}^{\infty} [n+1]_k z^{-(n+1)}\\ &= z^{-1}\sum_{n=-1}^{\infty} [n+1]_k z^{-n}.\end{aligned}$$

Because $[0]_k = 0 \times -1 \times \cdots \times (1-k) = 0$ we have

$$z\mathcal{Z}\big([n]_k u_n\big) = \sum_{n=0}^{\infty} [n+1]_k z^{-n}.$$

Now

$$\begin{aligned}(z-1)\mathcal{Z}\big([n]_k u_n\big) &= \sum_{n=0}^{\infty} [n+1]_k z^{-n} - \sum_{n=0}^{\infty} [n]_k u_n z^{-n}\\ &= \sum_{n=0}^{\infty} ([n+1]_k - [n]_k) z^{-n}.\end{aligned}$$

Observe that the falling factorial satisfies

$$\begin{aligned}[n+1]_k - [n]_k &= \big((n+1)n(n-1)\dots(n-k+2)\big) \quad - \quad \big(n(n-1)\dots(n-k+1)\big)\\ &= [n]_{k-1}(n+1-n+k-1)\\ &= [n]_{k-1}k\end{aligned}$$

and so

$$\begin{aligned}(z-1)\mathcal{Z}\big([n]_k u_n\big) &= k\sum_{n=0}^{\infty} [n]_{k-1} z^{-n}\\ &= k\mathcal{Z}\big([n]_{k-1} u_n\big).\end{aligned}$$

We obtain the following recursive equation for $\mathcal{Z}\big([n]_k u_n\big)$,

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k}{z-1}\mathcal{Z}\big([n]_{k-1} u_n\big).$$

Unravelling this recursion we obtain

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k}{z-1} \times \frac{k-1}{z-1} \times \frac{k-2}{z-1} \times \cdots \times \mathcal{Z}([n]_0 u_n).$$

By definition $[n]_0 = 1$ for all $n \in \mathbb{Z}$ and so $\mathcal{Z}([n]_0 u_n) = \mathcal{Z}(u_n) = z/(z-1)$ with region of convergence $|z| > 1$. Thus,

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1$$

as required.

6.5. Find the discrete impulse response of the discrete time system corresponding with the second order difference equation $c_n = d_n - ad_{n-1} - bd_{n-2}$.

6.6. Let d_n be a sequence satisfying $d_n = 2d_{n-1} + 2^{n+1}$ and suppose that $d_0 = 0$. Show that $d_n = 2^{n+1}n$ for $n = 1, 2, \dots$. **Solution:** In Section 6.6 we found the discrete time system H with discrete impulse response $h_n = 2^n u_n$ is such that the respone $y = Hx$ is input signal x satisfied the equation

$$x = y - 2T_P(y).$$

Suppose that the sample period is $P = 1$. The response of H to input signal is $x(t) = 2^{t+1}u(t)$ is

$$\begin{aligned} y = H(x) &= \sum_{n\in\mathbb{Z}} h_n T_n(x) \\ &= \sum_{n=0}^{\infty} 2^n 2^{t-n+1} u(t-n) \\ &= 2^{t+1} \lfloor t \rfloor u(t) \end{aligned}$$

Let c and d be sequences with elements

$$c_n = x(nP) = 2^{n+1}u_{n-1}, \qquad d_n = y(nP) = 2^{n+1}nu_n$$

and observe that $d_0 = 0$. By definition of H these sequences satisfy the difference equation

$$c_n = 2^{n+1}u_n = d_n - 2d_{n-1}$$

as required.

Let us now consider an alternative method of solution that applyies the z-transform directly to the difference equation

$$d_n - 2d_{n-1} = 2^{n+1}u_{n-1}.$$

Applying the z-transform to both sides and using the time shift property we have

$$\mathcal{Z}(d) - 2z^{-1}\mathcal{Z}(d) = \mathcal{Z}(2^{n+1}u_n) = \frac{4}{z-2} \qquad |z| > 2.$$

The z-transform of d is then

$$\mathcal{Z}(d) = \frac{4z}{(z-2)^2} \qquad |z| > 2.$$

The inverse z-transform is found by putting $k = 1$ and $a = 2$ in (6.5.2). We obtain $d_n = 2^{n+1}nu_n$. Observing that $d_n = 0$ we have a solution.

6.7. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ satisfies the recursive equation $d_0 = 0, d_1 = 1$, and $d_n = d_{n-1} + d_{n-2}$ for $n \geq 2$. Find a closed form expression for the nth Fibonacci number. **Solution:** Consider the recursive equation

$$d_n - d_{n-1} - d_{n-2} = \delta_n.$$

Applying z-transforms to both sides gives

$$\mathcal{Z}(d) - z^{-1}\mathcal{Z}(d) - z^{-2}\mathcal{Z}(d) = \mathcal{Z}(\delta) = 1.$$

The region of convergence is the whole complex plane. We have

$$\mathcal{Z}(d) = \frac{z^2}{z^2 - z - 1} = z\left(\frac{z}{z^2 - z - 1}\right)$$

Our formula for the inverse z-transformation (6.5.2) involves z on the numerator and so applying partial fractions to the term within the brackets will be convenient. The roots of $z^2 - z - 1$ are

$$a = \frac{1-\sqrt{5}}{2}, \qquad b = \frac{1+\sqrt{5}}{2}$$

and by partial fractions (see Exercise 4.9) we obtain.

$$\frac{z}{(z-a)(z-b)} = \frac{a}{(a-b)(z-a)} - \frac{b}{(a-b)(z-b)}$$

and so

$$\mathcal{Z}(d) = \frac{az}{(a-b)(z-a)} - \frac{bz}{(a-b)(z-b)}$$

Both of these terms are in the form of (6.5.2) when $k = 0$ and so

$$d_n = \frac{1}{a-b} a^n u_n + \frac{1}{b-a} b^n u_n = \frac{b^n - a^n}{\sqrt{5}} u_n$$

Observe that $d_0 = 0$ and that $d_1 = 1$ as a result of $b - a = 1$.

∗6.8. Show that a discrete time system is stable if and only if its discrete impulse response is absolutely summable.

∗6.9. Let f and g be absolutely summable sequences. Show that the discrete convolution $f * g$ is also absolutely summable.

∗6.10. Let H be a discrete time system with discrete impulse response h. The set $\operatorname{roc}_z h$ is defined as those complex numbers $z = e^{sP}$ such that $s = \operatorname{cep}\operatorname{dom}_P h$. Show that $\operatorname{roc}_z h$ is precisely the set of nonzero complex numbers such that the sequence $h_n z^{-n}$ is absolutely summable.

∗6.11. Let f, g, h be complex valued sequences such that

$$\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |f_k h_m g_{n-m-k}| < \infty.$$

Show that the discrete convolution is associative for these sequences. That is, show that $(f * g) * h = f * (g * h)$. **Solution:** Let f, g, h be sequences. We have

$$\begin{aligned}
((f*g)*h)_n &= \sum_{m\in\mathbb{Z}} h_m (f*g)_{n-m} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} g_k f_{n-m-k} \\
&= \sum_{m\in\mathbb{Z}} h_m \sum_{k\in\mathbb{Z}} f_k g_{n-m-k} \qquad \text{commutivity } f*g = g*f \\
&= \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} f_k h_m g_{n-m-k}
\end{aligned}$$

Under the assumptions stated on f, g, h Fubini's theorem [Rudin, 1986, Theorem 8.8] may be used to justify swapping the order of summation leading to

$$
\begin{aligned}
((f * g) * h)_n &= \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} f_k h_m g_{n-k-m} \\
&= \sum_{k\in\mathbb{Z}} f_k \sum_{m\in\mathbb{Z}} h_m g_{n-k-m} \\
&= \sum_{k\in\mathbb{Z}} f_k (g * h)_{n-k} \\
&= (f * (g * h))_n.
\end{aligned}
$$

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