The error performance of the A_n lattices

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Lattices

The Voronoi cell

The probability of coding error

The lattice A_n

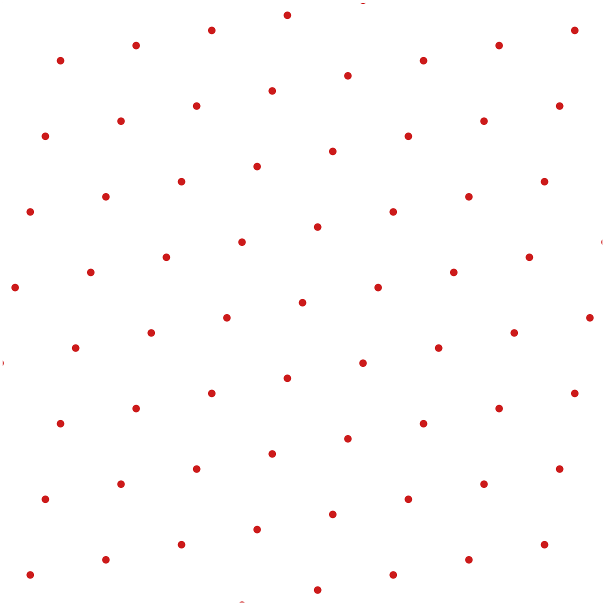


Figure : A lattice in 2 dimensions

Lattices

- A **lattice**, L, is a discrete set of points in $\mathbb{R}^m$ such that

$$L = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} = \mathbf{B}\mathbf{w}, \mathbf{w} \in \mathbb{Z}^n\}$$

- The matrix $\mathbf{B}$ is an $m \times n$ matrix called the **generator matrix**.
- Can write this more succinctly as

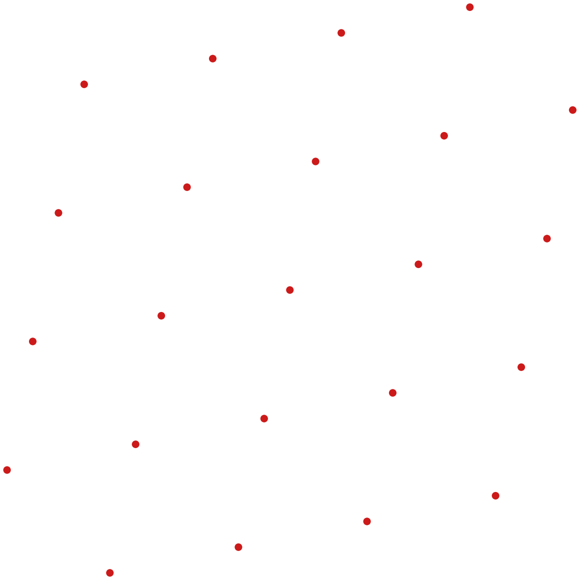
$$L = \mathbf{B}\mathbb{Z}^n$$

- $\mathbb{Z}^n$ is called the **integer lattice**.

The Voronoi Cell

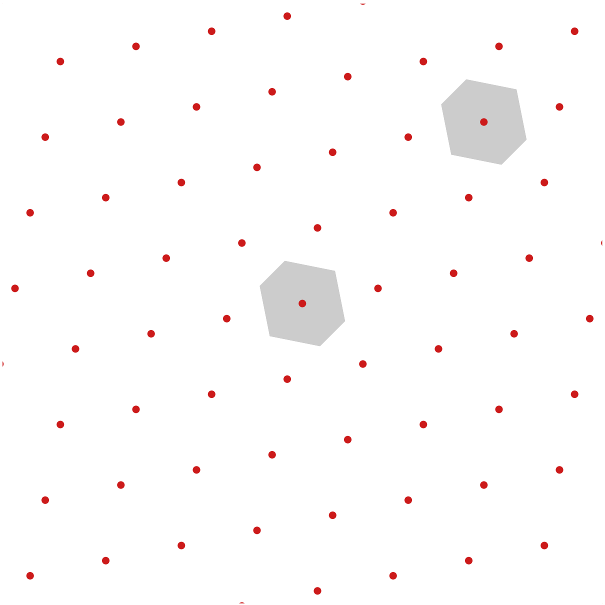
Definition

The **Voronoi cell** is the region that is closer to the origin than any other lattice point.



The Voronoi Cell

- The Voronoi cell is a convex polytope (an n-dimensional polygon).
- Translating the Voronoi cell by a lattice point $\mathbf{x}$ gives the region closer to $\mathbf{x}$ than any other lattice point.



The Voronoi Cell

The Voronoi cell encodes many interesting properties about a lattice.

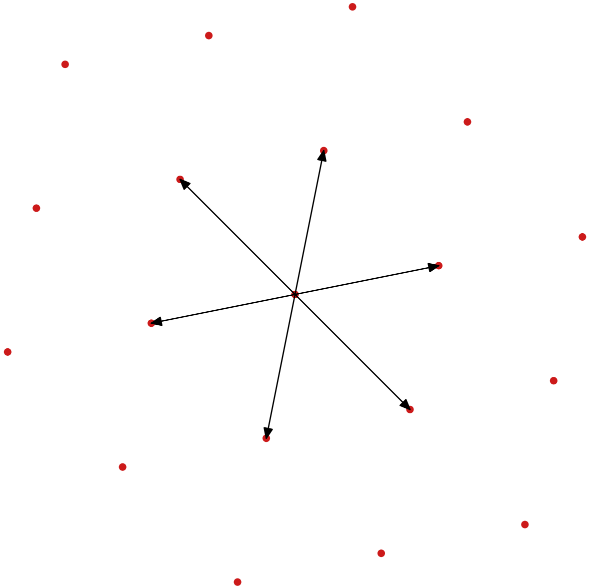


Figure : There are 6 relevant vectors

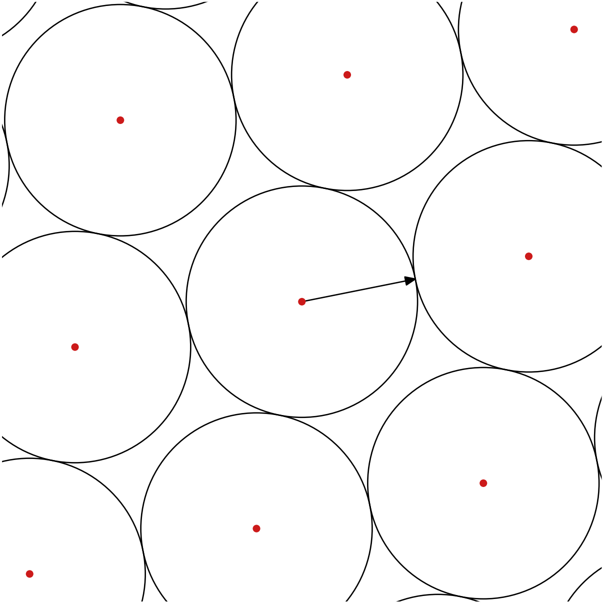


Figure : The packing radius

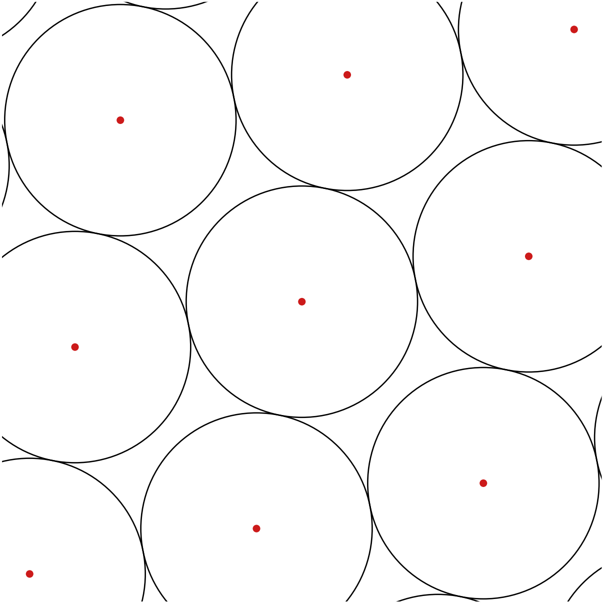


Figure : The kissing number is 4

The quantiser constant

- Consider using the points of a lattice Λ as a quantiser for a uniform source.
- The integral
$$D(\Lambda) = \int_{\mathrm{Vor}(\Lambda)} \|\mathbf{x}\|^2 d\mathbf{x}$$
tells us the average distortion of this quantiser.

Example

The Voronoi cell of the integer lattice $\mathbb{Z}^n$ is the hypercube

$$\left[-\frac{1}{2}, \frac{1}{2}\right)^n$$

and

$$D(\mathbb{Z}^n) = \int_{-1/2}^{1/2} \dots \int_{-1/2}^{1/2} \|\mathbf{x}\|^2 d\mathbf{x} = \frac{n}{12}$$

The quantiser constant

Conway and Sloane (1982) calculated D for the **root lattices**

$$\mathbb{Z}^n, \quad A_n, \quad D_n, \quad E_6, \quad E_7 \ \text{ and } \ E_8$$

and the dual lattices

$$A_n^* \ \text{ and } \ D_n^*$$

The quantiser constant

They obtain:

$$D(D_n) = \frac{n}{6} + \frac{n}{n+1}, \quad D(D_n^*) = \text{long formula!}$$

$$D(A_n) = \frac{n\sqrt{n+1}}{12} + \frac{n}{6\sqrt{(n+1)}}, \quad D(A_n^*) = \text{long formula!}$$

$$D(E_8) = \frac{929}{1620}, \;\; D(E_7) = \sqrt{2}\frac{163}{288}, \;\; D(E_6) = \sqrt{3}\frac{15}{28}$$

We know precisely how these lattices perform as quantisers.

The probability of coding error

- Consider using the points of a lattice Λ as a code for the Gaussian channel.
- The integral

$$P_C(\Lambda) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\mathrm{Vor}(\Lambda)} e^{-\|\mathbf{x}\|^2/2\sigma^2} d\mathbf{x}$$

  tells us the probability of correct decoding for this code.
- So we are interested in integrals of the form

$$\int_{\mathrm{Vor}(\Lambda)} e^{-\|\mathbf{x}\|^2} d\mathbf{x}$$

- Conway and Sloane (1982) hinted that computing such integrals might be possible.
- We are going to solve it for the lattice A_n.

The probability of coding error

- By expanding $e^x = 1 + x + \frac{x^2}{2} + \dots$ we obtain

$$P_C(\Lambda) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\mathrm{Vor}(\Lambda)} 1 - \frac{\|\mathbf{x}\|^2}{2\sigma^2} + \frac{\|\mathbf{x}\|^{2m}}{4\sigma^4 2!} - \dots d\mathbf{x}$$

$$= \frac{1}{(\sqrt{2\pi}\sigma)^n} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m \sigma^{2m} m!} \int_{\mathrm{Vor}(\Lambda)} \|\mathbf{x}\|^{2m} d\mathbf{x}.$$

- To obtain accurate approximations to the probability of error it is enough to know the values of

$$\int_{\mathrm{Vor}(\Lambda)} \|\mathbf{x}\|^{2m} d\mathbf{x}$$

  for sufficiently large m.
- We call these the **moments** of the Voronoi cell.

The lattice A_n

- The lattice A_n are all those points from $\mathbb{Z}^{n+1}$ with elements that sum to zero,

$$A_n = \left\{ \mathbf{x} \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\}$$

- A generator matrix for A_n is the $(n+1) \times n$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

The lattice A_n

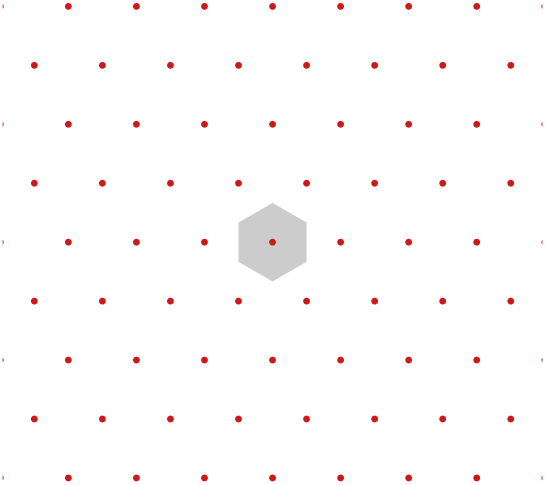


Figure : In 2 dimensions A_2 is the **hexagonal lattice**

The lattice A_n

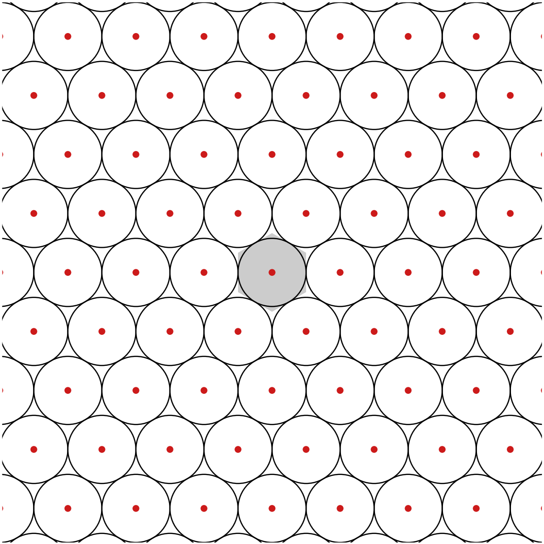


Figure : In 2 dimensions A_2 is the **hexagonal lattice**

The lattice A_n

- A_3 is called the **body centered cubic lattice**.
- It is the densest packing in 3 dimensions.
- We are interested in the moments for A_n,

$$C_n(m) = \int_{\mathrm{Vor}(A_n)} \|\mathbf{x}\|^{2m} d\mathbf{x}.$$

The lattice A_n

Theorem

The Voronoi cell of A_n is the projection of the $n + 1$ dimensional hypercube orthogonal to one of its vertices.

The lattice A_n

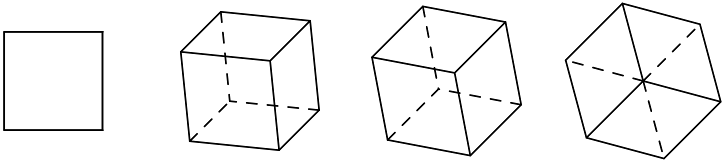


Figure : Orthogonal projection of a cube as it is rotated about its center

The lattice A_n

- Lattices with this property are called **zonotopal lattices**.
- Allows us to write integrals over $\mathrm{Vor}(A_n)$ as integrals over (almost) the $n+1$ dimensional hypercube.

The moments of the lattice A_n

The moment $C_n(m)$ is

$$C_n(m) = m! \frac{n\sqrt{n+1}}{n+2m} \sum_{k=0}^{m} \sum_{a=0}^{k} \sum_{b=0}^{k-a} \frac{G(n-1, a, 2k-2a-b)}{H(n,m,k,a,b)}$$

where the function

$$H(n,m,k,a,b) = \frac{(n+1)^{m-a} a!(m-k)!b!(k-a-b)!}{(-1)^{k-a} 2^b n^{m-k}}$$

and the function $G(n,c,d)$ satisfies the recursion

$$G(n,c,d) = \sum_{c'=0}^{c} \sum_{d'=0}^{d} \binom{c}{c'} \binom{d}{d'} \frac{G(n-1, c-c', d-d')}{2c'+d'+1}$$

with the initial conditions

$$G(1,c,d) = \frac{1}{2c+d+1} \qquad \text{and} \qquad G(n,0,0) = 1.$$

The moments of the lattice A_n

For fixed m it is possible to solve this recursion in n and obtain formula for the $C_m(n)$ in terms of n. The first four formula are:

$$C_0(n) = \sqrt{n+1} \qquad \text{the volume of } \mathrm{Vor}(A_n),$$

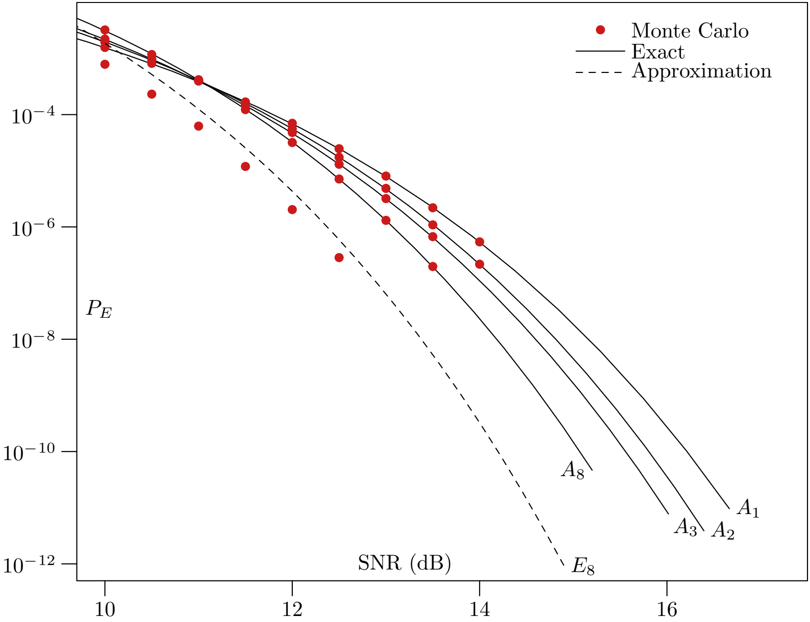
$$C_1(n) = \frac{n(n+3)}{12\sqrt{n+1}} \qquad \text{the second moment of } \mathrm{Vor}(A_n),$$

$$C_2(n) = \frac{50n + 55n^2 + 34n^3 + 5n^4}{720(1+n)^{3/2}},$$

$$C_3(n) = \frac{1960n + 2142n^2 + 2681n^3 + 1423n^4 + 399n^5 + 35n^6}{60480(1+n)^{5/2}},$$

$$\vdots$$

We have tabulated these formula for $m = 0$ to 40.



Monte Carlo
Exact
Approximation
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
P_E
A_8
A_1
A_3
A_2
E_8
SNR (dB)
10
12
14
16

What now?

- A_n does not produce good codes in large dimension.
- Can similar ideas be applied to other lattices or codes?